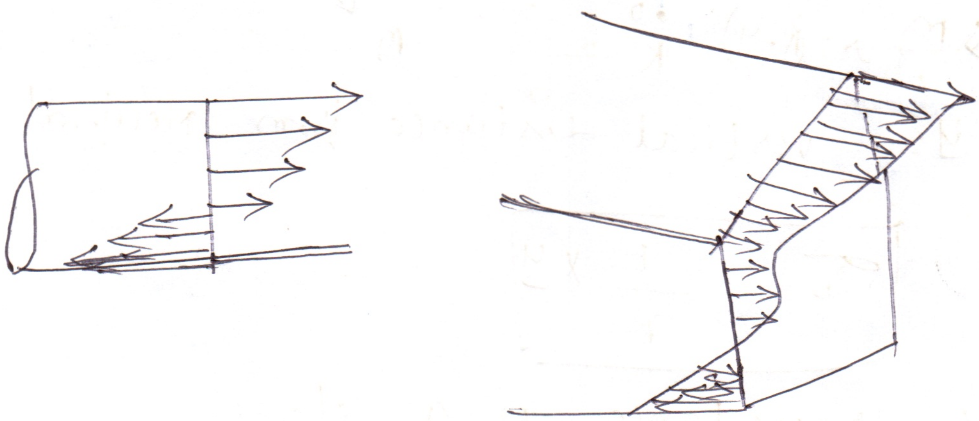
Moment of Inertia

Area Moments of inertia : Definition – Polar moment of inertia, Transfer theorem, Moments of inertia of Composite figures, products of inertia, Transfer formula for Product of inertia.

Mass moment of inertia :- Moment of inertia of masses, Transfer formula for mass moments of inertia, mass moment of inertia of Composite bodies

Introduction :- Previously we discuss how to determine the location of the resultant of distributed forces by using first moment of an Area, volume etc. Thus we saw the methods to determine the centroid, centre of mass and centre of gravity of various bodies. In this chapter, we will discuss the second moment of an area across a cross section, which finds application in the design of structural members such as beams, columns, etc.

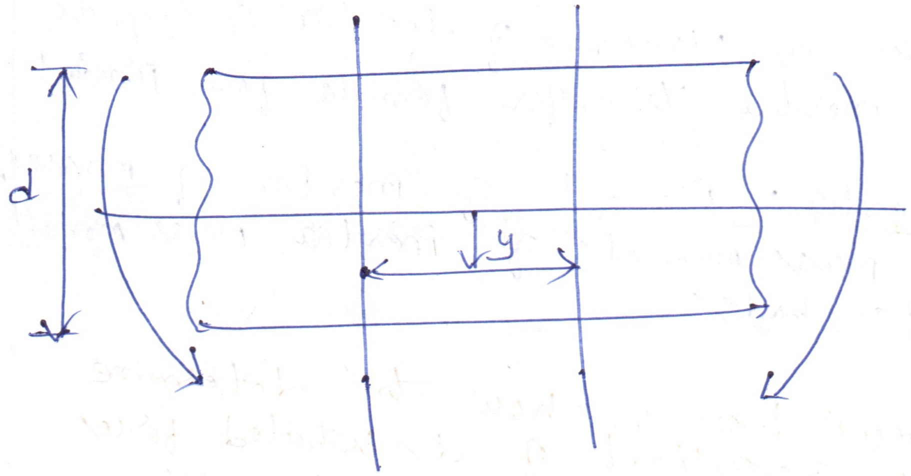


⇒ MR_{xx} → It is the moment of resistance of given c/s of the beam in N-mm
This is constant for any given c/s of the beam
– To avoid failure the applied bending moment "M" must be less than (MR_{xx}) $[M < MR_{xy}]$

$$dM_{R_{xx}} = \frac{E}{R} \times y^2 \, dA$$

→ I (or) → It is the second moment of area moment of inertia w.r.to neutral axis in (mm^4)

* Consider a member when it is loaded

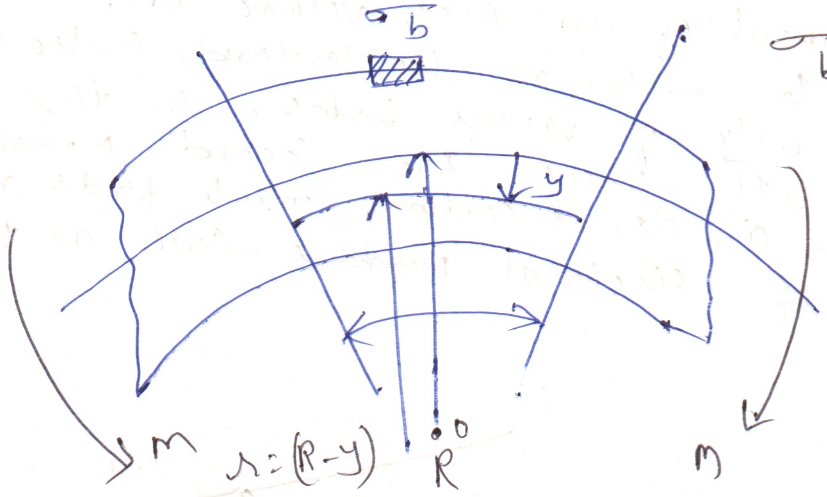


Hook's law

$$\sigma_b = E\,\epsilon_b$$

$$\sigma_b = \frac{E \times y}{R}$$

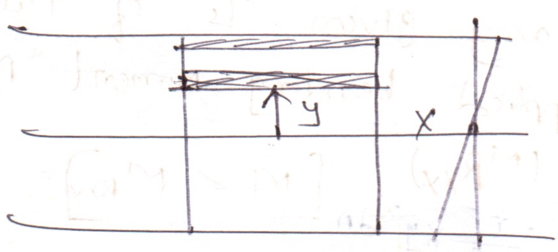
$$\sigma_b = \frac{E}{R} \times y$$



y = vertical distance from Neutral axis

$$\boxed{\sigma_b = \frac{E}{R} \times y}$$

Moment of resistance of a c/s



$\sigma_b = dF$

$\sigma_b = dF$

$M = F \times \perp$

$$dM_{R_{xx}} = dF \times y = \sigma_b \times y \times dA$$

$$dMR_{xx} = \frac{E}{R} y^2\, dA$$

$$MR_{xx} = \frac{E}{R}\int_A y^2 dA$$

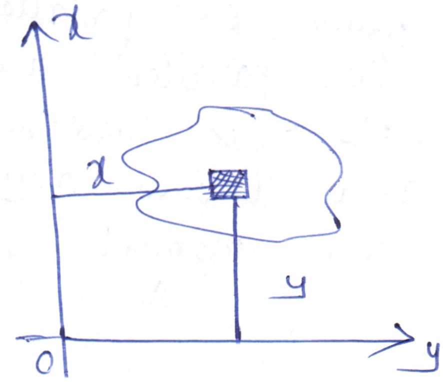
$$\frac{MR_{xx}}{I} = \frac{E}{R} \qquad \left[I = \int_A y^2 dA \right]$$

Second Moment of an Area (or) Moment of Inertia

→ Consider a plane lamina of area "A" as shown in fig. If we consider a small element of area dA at distance x & y from the origin

→ First moments with respect to x & y axis

$$dm_x = y\,dA \quad \& \quad dm_y = x\,dA$$



→ we take moment of first moment of the elemental area dA i.e

$$y(y\,dA) \quad \& \quad x(x\,dA)$$

→ Then it is called second moment of the elemental area dA about the respective axes

$$I_{xx} = \int y^2 dA, \quad I_{yy} = \int x^2 dA$$

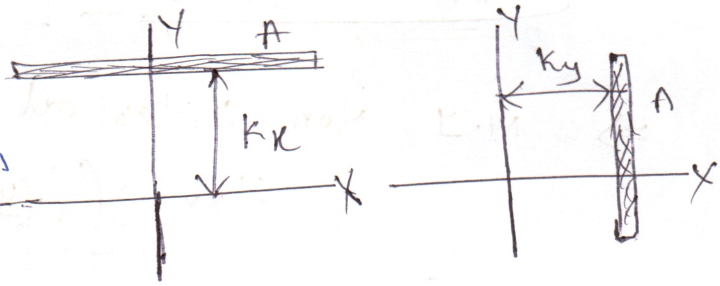
* Moment of an area — moment of inertia

It is always positive b<> square of the ⊥lar distance

Radius of gyration:-

If we concentrate the entire area "A" of the lamina into a thin strip parallel to the x-axis such at this distance k_x from the x-axis such that I_{xx} is the same for the area

$$\boxed{I_{xx} = A k_x^2}$$



this K_x = Radius of gyration

$$k_y = \sqrt{\frac{I_{yy}}{A}} \quad k_x = \sqrt{\frac{I_{xx}}{A}}$$

Moment of Inertia :-

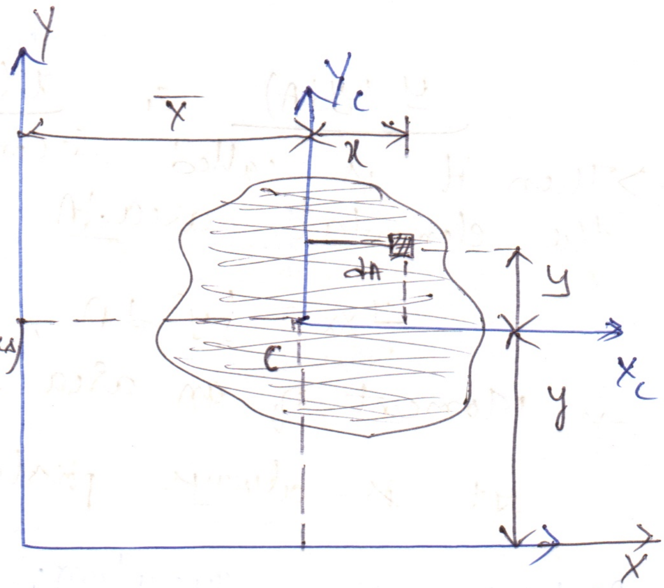
The property of a matter by virtue of ~~its~~ which it resists any change in its state of rest or uniform motion is called Inertia

Transfer formula (or) parallel axis theorem :-

⇒ where we have to determine the moment of inertia of a section about different axes

⇒ If we know the moment of inertia about a centroidal axes ~~by using~~ when we can determine moment of inertia about a non-centroidal axes which is parallel to the centroidal axis by using the parallel axis theorem, also called transfer formula

⇒ This theorem relates the moment of inertia of an area w.r.to any axis in the plane of the area to the moment of inertia w.r.to a parallel centroidal axis



⇒ lamina Area A
Centroidal axes about lamina area $(X_c - Y_c)$

⇒ Non centroidal axis (X-Y axes) but parallel to the centroidal axes

⇒ Small elemental area dA
Coordinates w.r.to (x, y) & with respect Non centroidal axes area $(x + \bar{x}, \ y + \bar{y})$

⇒ $\boxed{\bar{I}_{xx} = \int y^2 \, dA}$ [Bar sign indicates M.I about centroidal axes]

⇒ M.I Non centroidal axis

$$I_{xx} = \int \left((y + \bar{y}) \right)^2 dA$$

$$I_{xx} = \int (y+\bar{y})^2 \, dA$$

$$I_{xx} = \int y^2 dA + \int \bar{y}^2 dA + \int 2y\bar{y} \, dA$$

$$= \int y^2 dA + \bar{y}^2 \int dA + \underline{2\bar{y}\int y \, dA}$$

→ First moment of Area about its Centroidal axis so it is Zero

$$I_{xx} = \int y^2 dA + \bar{y}^2 \int dA$$

$$\therefore \boxed{I_{xx} = \bar{I}_{xx} + A(\bar{y})^2}$$

Similarly $\boxed{I_{yy} = \bar{I}_{yy} + A(\bar{x})^2}$

Def: M.I of an area about an axis in the plane of the area is equal to the moment of Inertia about the centroidal axis and parallel axis passing through the given axis plus the product of the area and the square of the distance between the parallel axis

Polar Moment of Inertia :-

The M.I of an area of a plane figure w.r.to an axis ⊥lar to [x-y plane & passing through a pole "O"] is called the polar M.I and is denoted I_0 (or) I_z

$$I_z = \int r^2 dA \quad \text{(or)} \quad r^2 = x^2 + y^2$$

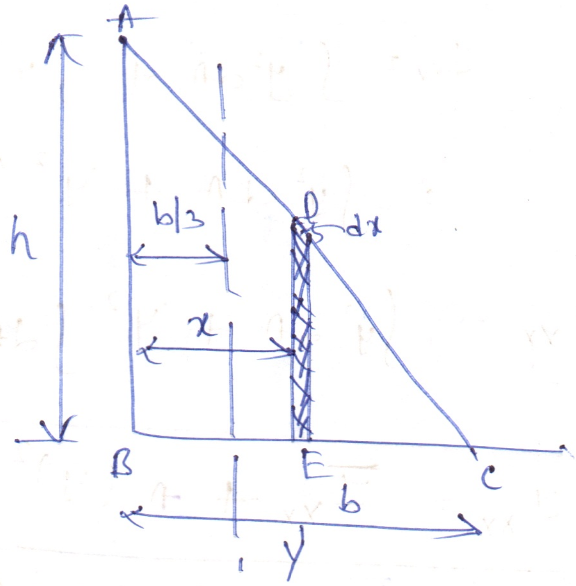
$$I_z = \int (x^2+y^2) \, dA$$

$$I_z = \int (x^2) \, dA + \int (y^2) \, dA$$

$$I_z = I_x + I_y$$

(1)

Determination :- I_{YY}



Small width dx, distance of x

$$dA = (DE)\,dx$$

$$\triangle ABC \cong \triangle DEC$$

$$\frac{AB}{DE} = \frac{BC}{EC}$$

$$DE = \frac{AB}{BC} \times EC \Rightarrow DE = \frac{h}{b}(b-x)$$

$$dA = \frac{h}{b}(b-x)\,dx$$

M.I small element

$$dI_{AB} = dA \cdot x^2 = \frac{h}{b}(b-x)dx \cdot x^2$$

$$dI_{AB} = \frac{hx^2}{b}(b-x)\,dx$$

$$dI_{AB} = \frac{h}{b}(bx^2 - x^3)\,dx$$

Total:

$$I_{AB} = \int dI_{AB} = \int_0^b \frac{h}{b}\left[bx^2 - x^3\right]dx$$

$$= \frac{h}{b}\left[b\frac{x^3}{3} - \frac{x^4}{4}\right]_0^b$$

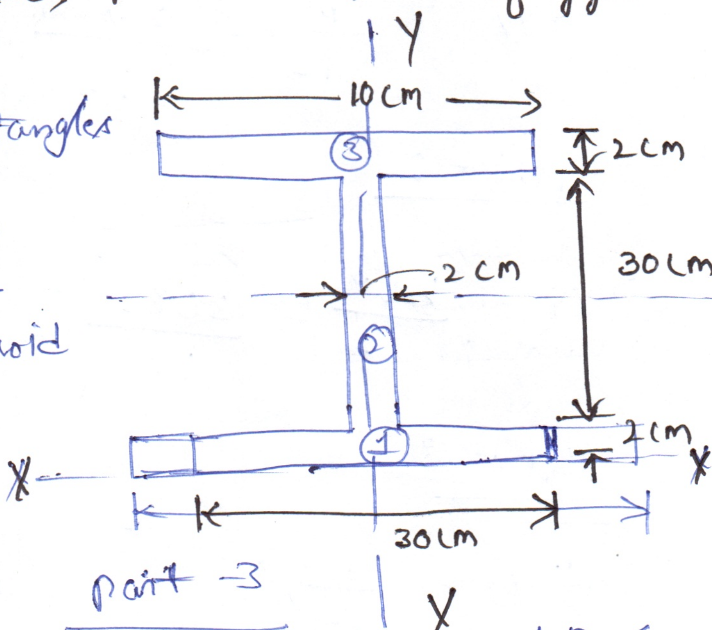
$$= \frac{1}{12}hb^3 \Rightarrow I_{AB} = I_{YY} + Ad^2$$

$$I_{AB} \Rightarrow \frac{1}{2}hb^3 = I_{YY} \cancel{+} + \frac{1}{2}bh\left(\frac{b}{3}\right)^2$$

$$I_{YY} = \frac{1}{36}hb^3$$

$$I_{YY} = \frac{1}{36}hb^3$$

(1) Find the M.I of the I-Section shown in fig about the centroidal axes. Also, find the radii of gyration about the same axes



Sol: It is made three rectangles

Due to symmetry, we know that $\bar{x} = 15$ cm

From the lower left corner

Then y-coordinate of the centroid is determined as follows

part - 1

$A_1 = 30 \times 2 = 60$ ($l \times b$)

$\bar{y}_1 = \frac{2}{2} = 1$

$a_1 y_1 = 60$ cm^3

part - 2

$A_2 = 30 \times 2 = 60$ ($l \times b$)

$\bar{y}_2 = \frac{2 + 30}{2} = \frac{32}{2}$

$= 2 + \frac{30}{2} = \frac{4 + 30}{2} = 17$

$a_2 y_2 = 1020$ cm^3

part - 3

$A_3 = 10 \times 2 = 20$ $\quad \Sigma = 140$

$y_3 = 2 + 30 + 1 = 33$

$2 + 30 + \frac{2}{2} = 33$

$a_3 y_3 = 660$ cm^3

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = 12.43 \text{ cm}$$

Moment of Inertia (M.I)

Part - 1

$I_{xx} = I_{xx} + A(\bar{y})^2$

$I_{xx} = \frac{lb^3}{12} = \frac{30 \times (2)^3}{12} = \frac{240}{12} = 120$

part 2

$I_{xx_2} = \frac{lb^3}{12} = \frac{2 \times 30^3}{12}$

$I_{2xx} = 4500$

$I_{xx3} = \frac{lb^3}{12} = \frac{10 \times 2^3}{12} = 6.67$

4526.67

part - 1

$$I_{yy} = \frac{bl^3}{12} = \frac{2 \times 30^3}{12} = 4500 \Rightarrow A_i(\bar{y}_i - y)^2$$

$$60(1 - 12.43)^2 = 7838.69\,cm^4$$

part - 2

$$I_{yy} = \frac{bl^3}{12} = \frac{30 \times 2^3}{12} = 20 \Rightarrow A_1(\bar{y}_1 - \bar{y})^2$$

$$60(17 - 12.43)^2 = 1253.09\,cm^4$$

part - 3

$$I_{yy} = \left(\frac{1}{12}\right) \times 2 \times 10^3 = 166.67 \Rightarrow 20(33 - 12.43)^2 =$$

$$= 8462.5\,cm^4$$

$$4686.67 \qquad 17554.28$$

$$\Rightarrow A(\bar{x}_1 - \bar{x})^2\ cm^4$$

$$\Rightarrow 60(15 - 15) = 0,\ 0, 0$$

$$\Rightarrow I_{xx} = \Sigma I_{xy} + \Sigma A(\bar{y}_i - \bar{y}_1)^2$$

$$I_{xy} = 4526.67 + 17554.28 = \underline{22080.95\,cm^4}$$

$$I_{yy} = \Sigma I_{yy} + \Sigma A(\bar{x}_i - \bar{x})^2$$

$$I_{yy} = 4686.67 + 0 = \underline{4686.67\,cm^4}$$

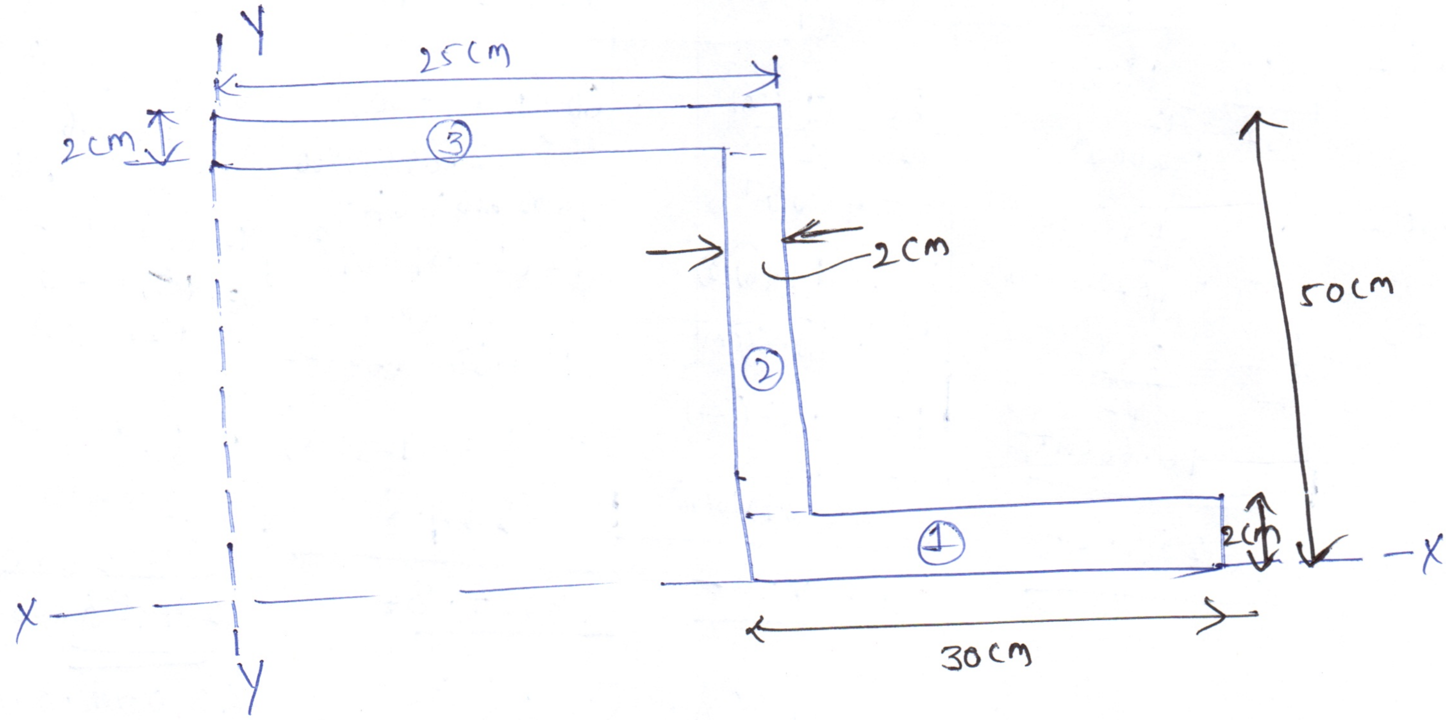
Radius of gyration:-

$$K_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$= \sqrt{\frac{22080.95}{140}} = 12.56\,cm$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{4686.67}{140}} = \underline{5.79\,cm}$$

(1) Determine M.I of the following Z - Section



S.NO	Element	A_i in cm^2	X_i in cm
part: 1	Rectangle 1	$a_1 = l \times b = 30 \times 2 = 60 cm^2$ ~~$x_1 = 23 + \frac{l}{2} = 23 + \frac{30}{2}$~~ ~~$x_1 = 38 cm$~~	$X_1 = 23 + l/2 = 23 + \frac{30}{2} = 38 cm$
part: 2	Rectangle 2	$a_2 = lb = 2 \times 46 = 92 cm^2$	$X_2 = 23 + \frac{l}{2} = 23 + \frac{2}{2} = 24 cm$
part: 3	Rectangle 3	$a_3 = lb = 25 \times 2 = 50 cm^2$	$X_3 = l/2 = \frac{25}{2} = 12.5 cm$
		202	

Y_i in cm

$y_1 = \frac{b}{2} = \frac{2}{2} = 1 cm$

$y_2 = 2 + \frac{b}{2} = 2 + \frac{46}{2} = 25 cm$

$y_3 = 48 + \frac{b}{2} = 48 + \frac{2}{2} = 49 cm$

$A_i X_i$ (cm^3)	$A_i Y_i$
$a_1 x_1 = 2280 cm^3$	$a_1 y_1 = 60 \times 1 = 60 cm^3$
$a_2 x_2 = 2208 cm^3$	$a_2 y_2 = 92 \times 25 = 2300 cm^3$
$a_3 x_3 = 12.5 cm^3$	$a_3 y_3 = 50 \times 40 = 2450 cm^3$

$$\bar{x} = \frac{A_1 X_1 + A_2 X_2 + A_3 X_3}{A_1 + A_2 + A_3} = 25.31 cm$$

$$\bar{y} = 23.81 cm$$

M.I :-

	$(I_{xx})_i$; cm^4	$(I_{yy})_i$; cm^4	$A_i(\bar{X}_i - X)^2$ cm^4	$A_i(\bar{Y}_i - Y)_i^2$ cm^4
Part 1	$\frac{lb^3}{12} = \frac{1(30)(2)^3}{12} = 20\,cm^4$	$\frac{bl^3}{12} = \frac{(2)(30)^3}{12} = 4500\,cm^4$	$60(25.31-38)^2 = 12.69\,cm \times 60$ 9662.16 $\cancel{161.4}$	$60(23.81-1)^2 = \cancel{13686}$ 31217.7666
Part 2	$\frac{lb^3}{12} = \frac{1}{12}(2)(46)^3 = 16222.66\,cm^4$	$\frac{bl^3}{12} = \frac{1}{12}(46)(2)^3 = 30.66\,cm^4$	$92(25.31-24)^2 = 1.31\,cm \times 92$ 157.88 $\cancel{120.52}$	$92(23.81-25)^2 = \cancel{130.28}$ 130.28
Part 3	$\frac{lb^3}{12} = \frac{25(2)^3}{12} = 16.66\,cm^4$	$\frac{bl^3}{12} = \frac{1}{12}(2)(25)^3 = 2604.16\,cm^4$	$50(25.31-12.5)^2 = 8204.8$	$50(49-23.81)^2 = \cancel{1259.5}$ $\cancel{31726.805}$ $\cancel{2758.38}$
	16259.32	7134.82	18024.84	$63,074.851$

$$I_{xx} = \Sigma I_{xx} + \Sigma A_i(\bar{Y}_i - Y)^2$$

$$I_{xx} = 16259.32 + 2758.38 = 79334.41\ cm^4$$

$$\Rightarrow I_{yy} = \Sigma I_{yy} + \Sigma A_i(\bar{X}_i - X_i)^2$$

$$I_{yy} = 7134.82 + 18024.84 = 25159.66\,cm^4$$

Radius of gyration

$$K_x = \sqrt{\frac{I_x}{A}}$$

$$K_x = \sqrt{\frac{79334.41}{202}} = 19.81\ cm$$

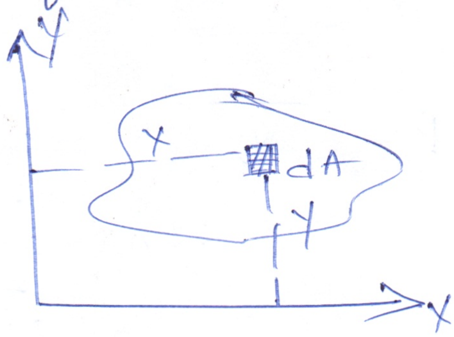
$$K_y = \sqrt{\frac{25159.66}{202}}$$

$$K_y = 11.16\ cm$$

Product of Inertia :-

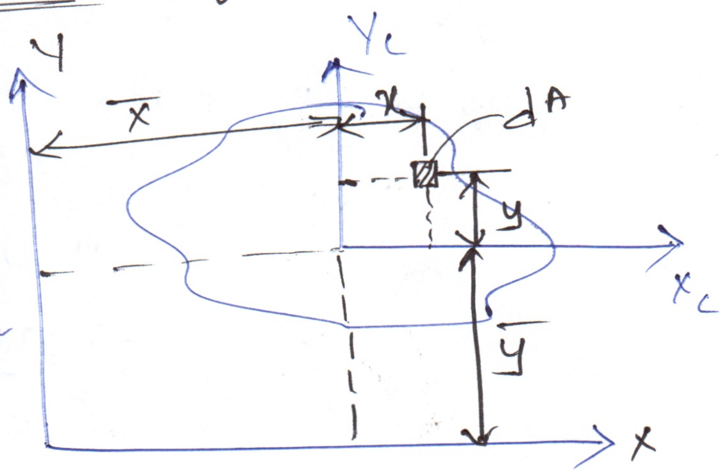
While finding the M.I of an area, we multiply each element of the area by the square of its distance from the axis. Thus However, if we multiply each element of the area by the product of its coordinates, i.e., $xy\,dA$ then it is called product of inertia

Hence the product of inertia of entire area is given as



$$I_{xy} = \int xy\,dA$$

Transfer theorem for product of Inertia :



$$I_{xy} = \int xy\,dA$$

$$I_{xy} = \int (x+\bar{x})(y+\bar{y})\,dA$$

$$= \int xy\,dA + \bar{y}\int x\,dA \Big\} \rightarrow \text{fis Mom}$$

$$+ \bar{x}\int y\,dA + \bar{x}\bar{y}\int dA$$

$$I_{xy} = I_{xy} + A\bar{x}\bar{y}$$

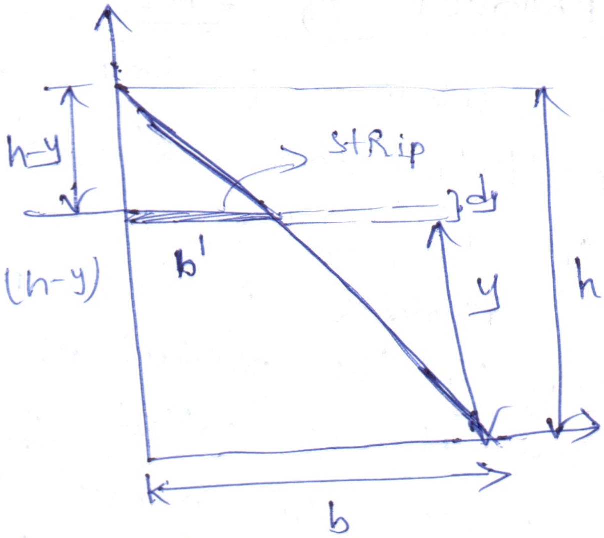
$$\boxed{I_{xy} = I_{xy} + A\,\bar{x}\cdot\bar{y}}$$

P.I Right angle triangle

$dA = b' dy$

Similar triangles

$$\frac{b'}{h-y} = \frac{b}{h} \Rightarrow b' = \frac{b}{h}(h-y)$$



$$\therefore \quad dA = \frac{b}{h}(h-y)\,dy$$

P.I of the strip about x-y axes is

$$dI_{xy} = \left[\frac{b'}{2}\right] y\, dA$$

$$= \left[\frac{b'}{2}\right] y \left[\frac{b}{h}(h-y)dy\right]$$

$$dI_{xx} = \frac{1}{2}\frac{b^2}{h^2}(h-y)^2\, y\, dy$$

$\therefore$ Therefore P.I of Right angle triangle

$$I_{xy} = \frac{1}{2}\frac{b^2}{h^2}\int (h-y)^2\, y\, dy$$

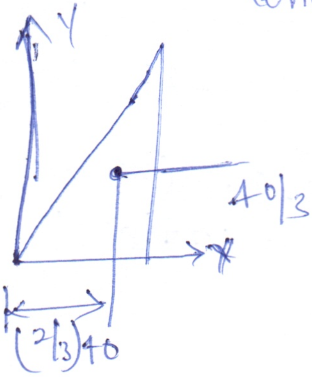
$$I_{xy} = \frac{1}{2}\frac{b^2}{h^2}\left[\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3}\right]_0^h$$

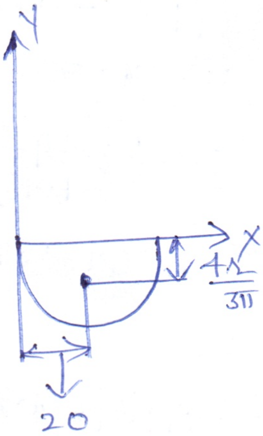
$$I_{xy} = \frac{b^2h^2}{24}$$

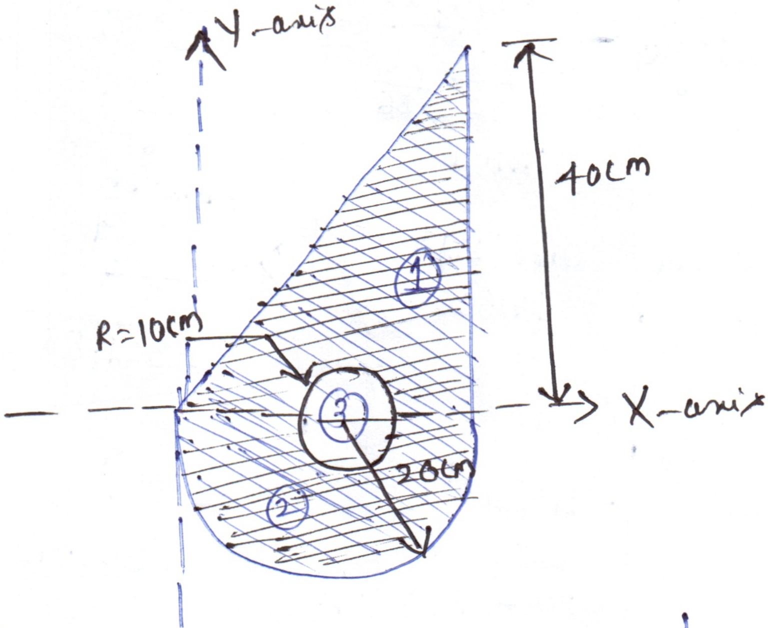
$$\bar{I}_{xy} = I_{xy} - A\bar{X}\bar{Y}$$

$$= \frac{b^2h^2}{24} - \frac{bh}{2}\left(\frac{b}{3}\right)\left(\frac{h}{3}\right) = \frac{-b^2h^2}{72}$$

③ Find the M.I of the shaded Area about the centroidal axes







	$A_i\ (cm)^2$	$x_i\ (cm)$	$y_i\ (cm)$	$A_i x_i\ (cm^3)$	$A_i y_i\ (cm^3)$
triangle	$\frac{1}{2} \times b \times h$ $= \frac{1}{2} \times 40 \times 40 = 800\,cm^2$	$\frac{2}{3}(40) = 26.67$	$\frac{1}{3}(40) = 13.33$	21336	10664
semicircle	$\frac{\pi r^2}{2} = \frac{\pi (20)^2}{2} = 628.32$	20	$\frac{-4(20)}{3\pi} = -8.49$	12566.4	-5334.44
circle	$-\pi r^2 = -\pi (10)^2$ $= -314.16$	20	0	-6283.2	0
	1114.16			27619.2	5329.56

$$\bar{X} = \frac{\Sigma A_i \bar{x}_i}{\Sigma A_i} = \underline{24.79\,cm}$$

$$\bar{Y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = \underline{4.78\,cm}$$

M.I :

	$(I_{xx})_i$ cm^4	$(I_{yy})_i$ cm^4	$A_i(\bar{y}_i-\bar{y})^2$ cm^4	$A_i(\bar{x}_i-\bar{x})^2$ cm^4
Triangle	$\frac{40\times(40)^3}{36} = \frac{bh^3}{36}$ $= 71111.11$	$\Rightarrow \frac{40\times40^3}{36}$ $= 71111.11$	$800(13.33-4.78)^2$ $=58482$	$800(26.67-24.79)^2$ $=2827.52$
Semi circle	$0.11(20)^4$ $= 17600$	$\Rightarrow \frac{\pi(20)^4}{8}$ $= 62831.85$	$628.32(-8.49-4.78)^2$ $= 110642.69$	$628.32(20-24.79)^2$ $=14416.24$
Circle	$-\pi(10)^4/4$ $= -7853.98$	$\Rightarrow -\frac{\pi(10)^4}{4}$ $= -7853.98$	$-314.16(0-4.78)^2$ $=-7178.05$	$-314.16(20-24.79)^2$ $=-7208.12$
	Σ 80857.13	Σ 126088.98	Σ 161946.64	Σ 10035.64

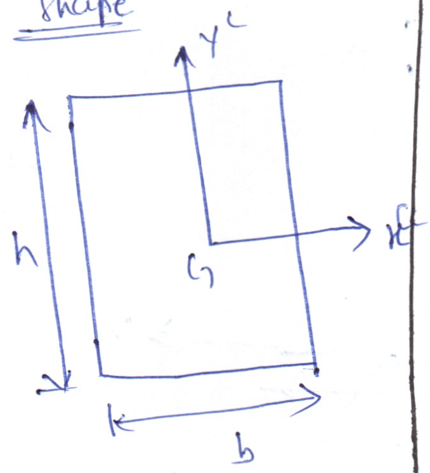
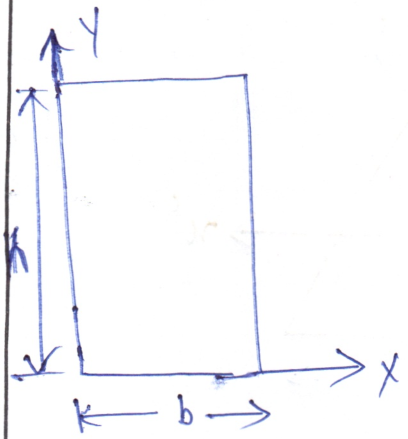
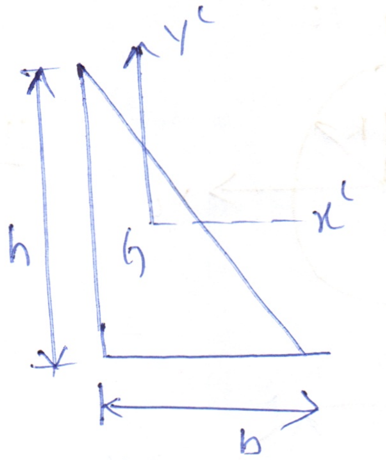
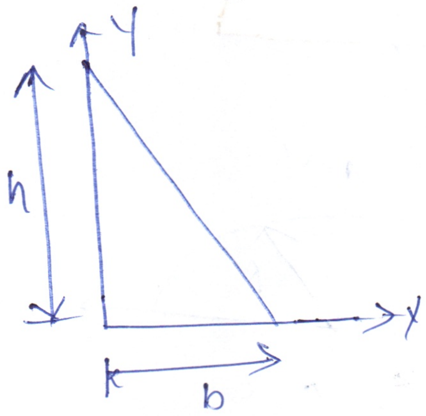
$$\bar{I}_{xy} = \Sigma(\bar{I}_{xx})_i + \Sigma A_i(\bar{y}_i-\bar{y})^2$$

$$= 80857.13 + 161946.64 = \underline{242803.77\ cm^4}$$

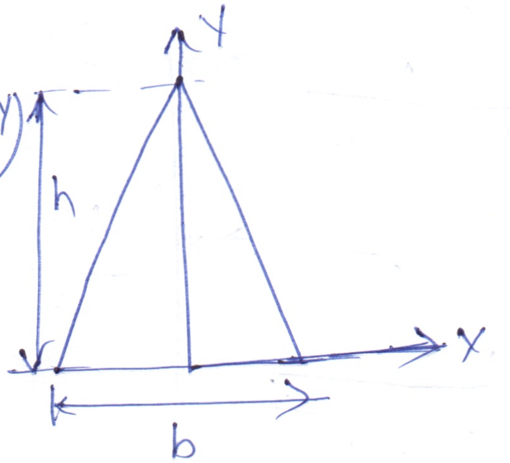
$$\bar{I}_{yx} = \Sigma(\bar{I}_{yy})_i + \Sigma A_i(\bar{x}_i-\bar{x})^2$$

$$= 126088.98 + 10035.64 = \underline{136124.62\ cm^4}$$

M.I of Regular shapes :-

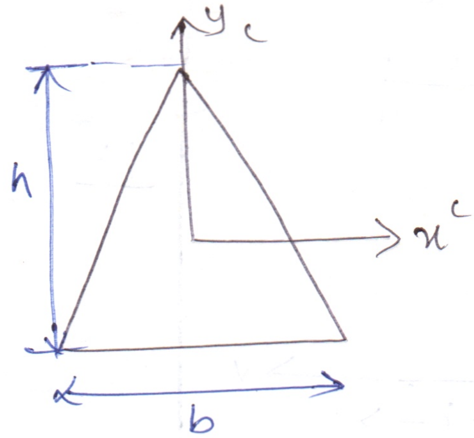
Name	Shape	I_{xx}	I_{yy}	I_{xy}
⇒ Rectangle (About centroidal axes)		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	0
⇒ Rectangle (About axes along the sides)		$\frac{bh^3}{3}$	$\frac{hb^3}{3}$	$\frac{b^2h^2}{4}$
⇒ Right angle triangle (About centroidal axes)		$\frac{bh^3}{36}$	$\frac{hb^3}{36}$	$-\frac{b^2h^2}{72}$
Right triangle (About X-Y axis)		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	$\frac{b^2h^2}{24}$

Issosceles triangle (About X-Y) axes



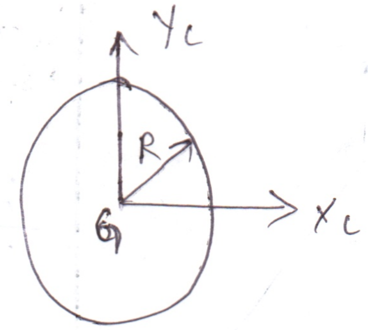
$\frac{bh^3}{12}$ $\frac{hb^3}{48}$ 0

Issosceles triangle (About centroidal Axes)



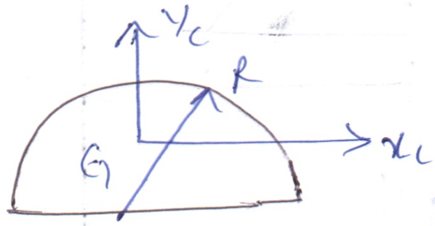
$\frac{bh^3}{36}$ $\frac{hb^3}{48}$ 0

Circle (About Centroidal axes)



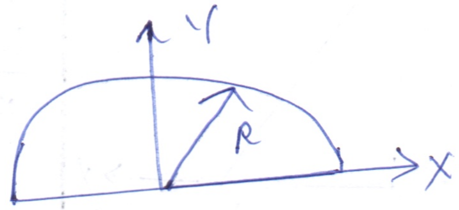
$\frac{\pi R^4}{4}$ $\frac{\pi R^4}{4}$ 0

Semi circle (About Centroidal Axes)



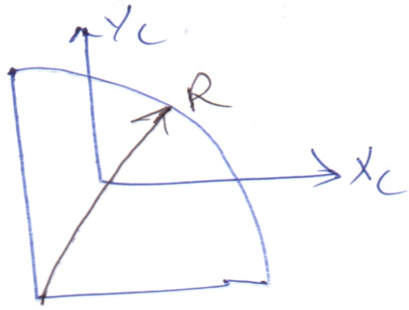
$0.11R^4$ $\frac{\pi R^4}{8}$ 0

Semi circle (About diametric Axes)



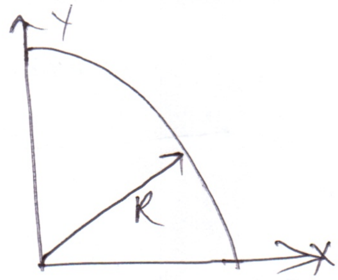
$\frac{\pi R^4}{8}$ $\frac{\pi R^4}{8}$ 0

Quarter circle About Centroidal axes



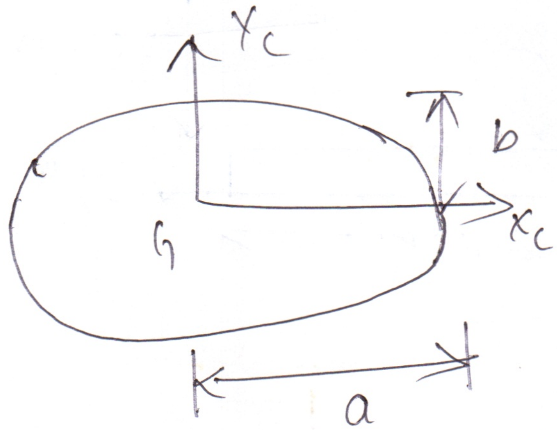
$0.055R^4$ $0.055R^4$ $-0.016R^4$

Quarter circle
(About X-Y axes)



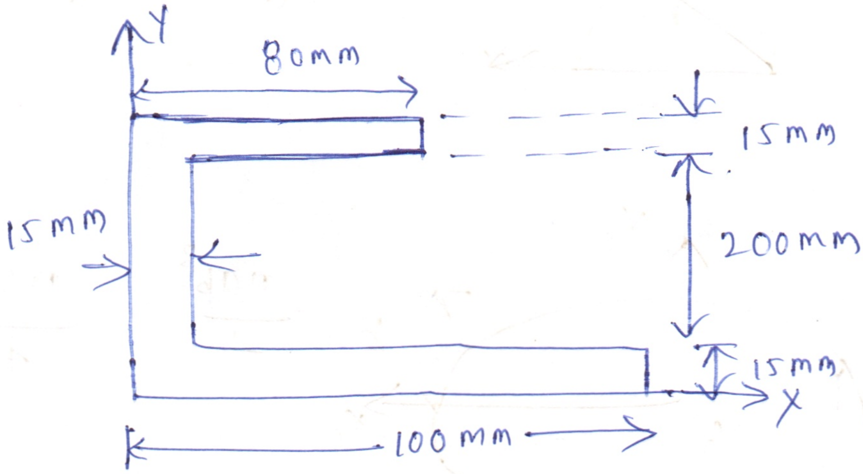
$\frac{\pi R^4}{16}$ $\frac{\pi R^4}{16}$ $\frac{R^4}{8}$

Ellipse
(About centroidal axes)

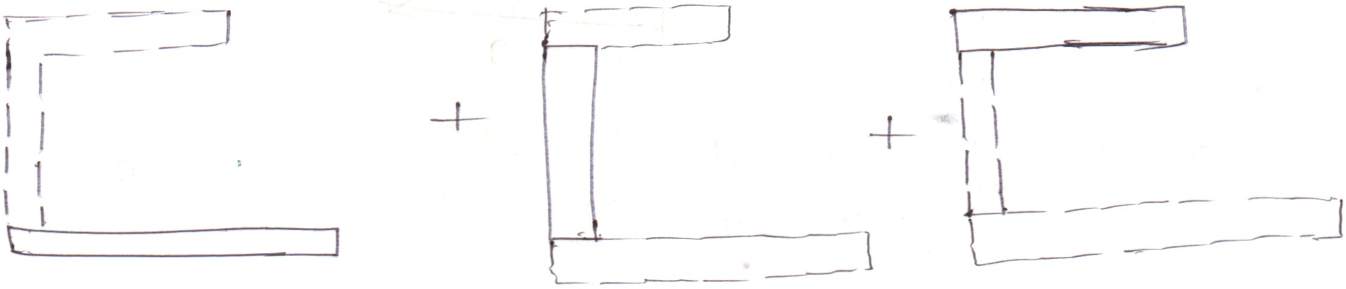


$\frac{\pi a b^3}{4}$ $\frac{\pi b a^3}{4}$ 0

1) find the P.I of the channel Section shown w.r.to Centroidal axes



Sol:-



S.NO & Element	$A_i\ (mm)^2$	X_i (mm)	$\bar{y}_i$ (mm)	$A_i x_i\ (mm)^3$	$A_i \bar{y}_i\ (mm^3)$
Rectangle 1	100×15 l×b = 1500	50 = l/2	$\frac{15}{2} = 7.5$	75000	11250
Rectangle 2	15×200 l×b = 3000	7.5 = l/2	$15+\left(\frac{200}{2}\right) = 115$	22500	134500
Rectangle 3	80×15 l×b = 1200	40 = l/2	$215+\left(\frac{15}{2}\right) = 222.5$	48000	267000
				$\Sigma A_i x_i$ 145500	$\Sigma A_i \bar{y}_i$ 623250

$\Sigma A_i = 5700$

$$\bar{X} = \frac{\Sigma A_i \bar{x}_i}{\Sigma A_i} = 25.53\,mm \qquad \bar{y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = 109.34\,mm$$

P.I	$(I_{xy})_i$ mm⁴	$A_i(\bar{x}_i - \bar{X})(\bar{y}_i - \bar{Y})$ mm⁴
1	0	$1500(50-25.53)(7.5-109.34) = -3738037.2$
2	0	$3000(7.5-25.53)(115-109.34) = -306149.4$
3	0	$1200(40-25.53)(222.5-109.34) = 1964910.2$

$$\Sigma A_i(\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y}) = -2079276.4 \ mm^4$$

$I_{xy} = (I_{xy})_i + \Sigma A_i(\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y}) =$

Mass moment of Inertia :-

mass m.I of a solid gives the ability of the solid to oppose any change in rotational motion about a specific axis

The larger the mass m.I the smaller the angular acceleration about that axis for a given torque.

In this chapter we will discuss the concepts related to mass moment of Inertia. we can easily define and understand the topics of mass m.I by relating them with Area m.I topics

Mass Moment of Inertia :-

Consider a body of mass M. If we take an infinitesimally small element of mass dm then its mass m.I about any axis is defined as the product of the mass dm and the square of the ~~thus~~ distance from the axis. Hence, its mass m.I about the z-axis is

$$\Rightarrow dI_{zz} = r^2\, dm$$

Integrating the above expression

$$I_{zz} = \int r^2\, dm$$

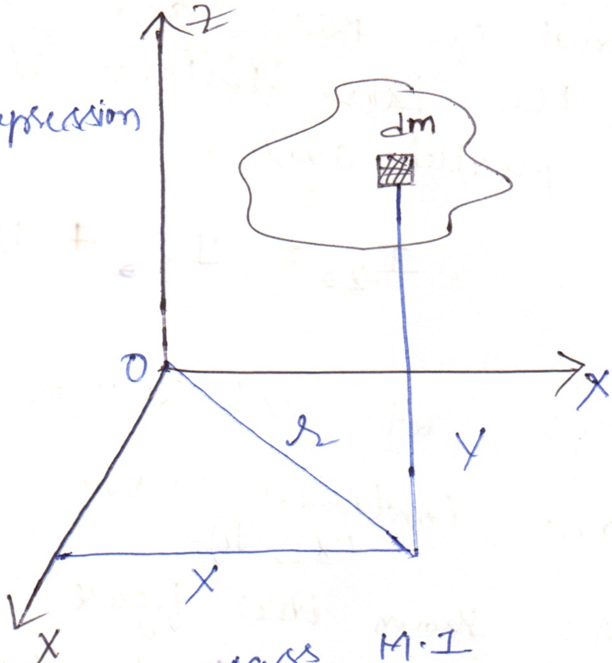
$$\therefore r^2 = x^2 + y^2$$

$$I_{zz} = \int (x^2+y^2)\, dm$$

Similarly

$$I_{xx} = \int (y^2+z^2)\, dm$$

$$I_{yy} = \int (x^2+z^2)\, dm$$



Mass M.I

Its units is :- kg.m^2

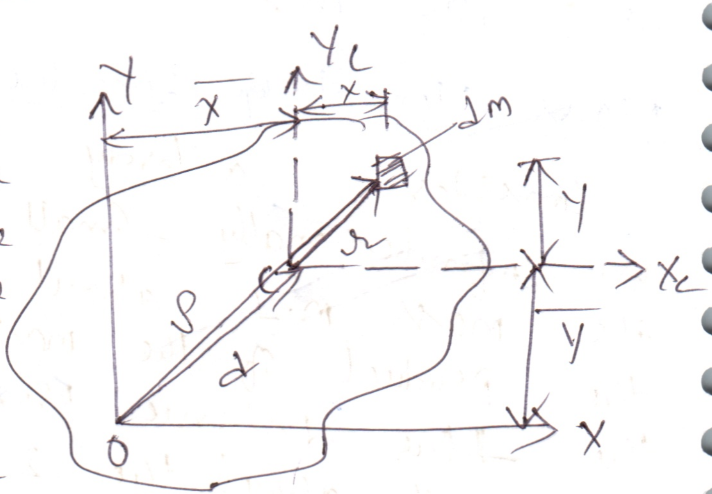
Radius of gyration :-

The radius of gyration is defined as the distance from the axis of inertia to the point at which the entire mass "m" of the body may be assumed to be concentrated and still have the same moment of inertia

$$k = \sqrt{\frac{I}{m}}$$

For composite bodies $\Rightarrow k = \sqrt{\frac{I_1 + I_2 + I_3 - -}{m_1 + m_2 + m_3 - -}}$

TRANSFER THEOREM (or) Transfer formula (or) parallel axis theorem :-



Theorem : The mass m.I of a body about an axis at a distance 'd' and parallel to the centroidal axis is equal to the sum of moment of inertia about the centroidal axis and product of mass and square of the distance b/w the of the ⊥lar parallel axis

$$I_{zz} = \bar{I}_{zz} + Md^2$$

Proof :- let us consider two sets of reference axes one centroidal (x_c, y_c) & other non centroidal (x, y) axes

As shown the figure the cross section of the mass such that the z-axis is ⊥lar to the plane of the paper. If we take a small element of mass dm then its coordinates w.r.to the centroidal axes are (x_c, y_c) and w.r.to non centroidal axes are $[(x+\bar{x}), (y+\bar{y})]$, where $\bar{x}$ & $\bar{y}$ are coordinates of the centroid (C)

$$\rho^2 = (x+\bar{x})^2 + (y+\bar{y})^2$$

$$= x^2 + \bar{x}^2 + 2x\bar{x} + y^2 + \bar{y}^2 + 2y\bar{y}$$

$$\Rightarrow \rho^2 = (x^2+y^2) + (\bar{x}^2+\bar{y}^2) + 2x\bar{x} + 2y\bar{y}$$

$$(x^2+y^2) = r^2 \quad \& \quad (\bar{x}^2+\bar{y}^2) = d^2$$

$$\rho = r^2 + d^2 + 2x\bar{x} + 2y\bar{y}$$

Mass M.I About Non centroidal z-axis

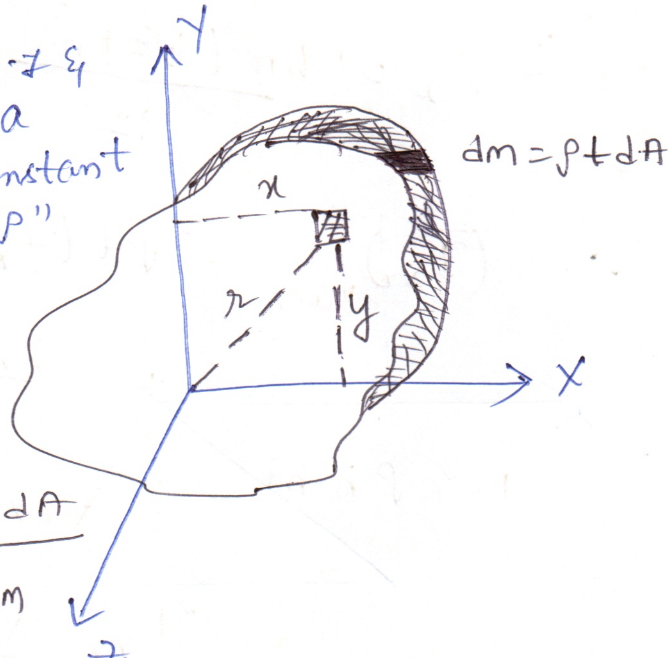
$$I_{zz} = \int \rho^2 dm = \int r^2 dm + \int d^2 dm + 2\bar{x}\underbrace{\int x\,dm}_{\text{first M.I}}$$

$$+ 2\bar{y}\underbrace{\int y\,dm}$$

$$\therefore \boxed{I_{zz} = \bar{I}_{zz} + md^2}$$

Mass M.I of Thin plates :-

The relationship b/w Area M.I & mass M.I & we considered a thin homogeneous plate with constant thickness "t" and mass density "ρ" Assume the plate to be thus to the z-axis.



If we take an Infinitely a small element of mass $dm = \rho t dA$ then M.I w.r.to x-axis is $y^2 dm$

Then Mass M.I of entire plate is

$$I_{xx} = \int y^2 dm = \rho t \int y^2 dA \quad \left[\therefore \int y^2 dA = I_{xx}\right) \text{ Area M.I}$$

$$(I_{xx})_{mass} = \rho t\,(I_{xx})_{Area}$$

$$\boxed{(I_{xx})_{mass} = \rho t\,(I_{xx})_{Area}}$$

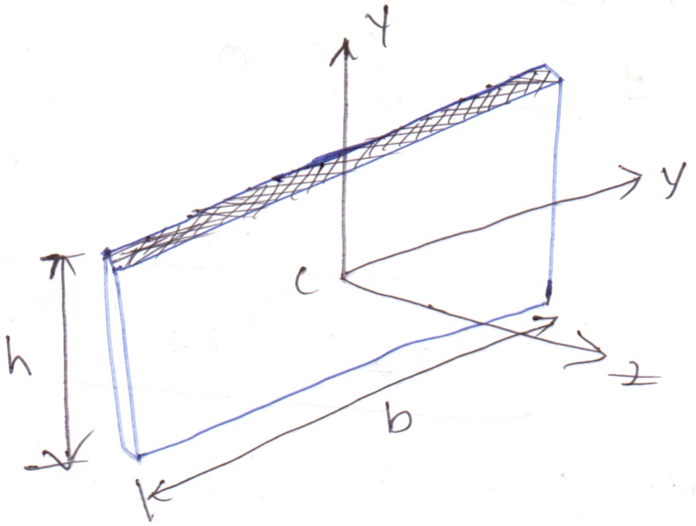
Similarly

$$\boxed{(I_{YY})_{mass} = \rho t \, (I_{YY})_{Area}}$$

$$\Rightarrow I_{zz} = \int r^2 dm = \rho t \int (x^2 + y^2) \, dA$$

$$\boxed{(I_{zz})_{mass} = \rho t \, (I_{zz})_{Area}}$$

Mass m.I of a thin rectangular plate :-



$\Rightarrow M = \rho b h t$

$$\Rightarrow I_{XX} = \frac{bh^3}{12}$$

$$I_{YY} = \frac{hb^3}{12}$$

$$I_{zz} = \frac{bh(b^2 + h^2)}{12}$$

[Acc. to polar m.I
$I_{zz} = I_{XX} + I_{YY}$]

$$\therefore (I_{XX})_{mass} = \rho t (I_{XX})_{Area}$$

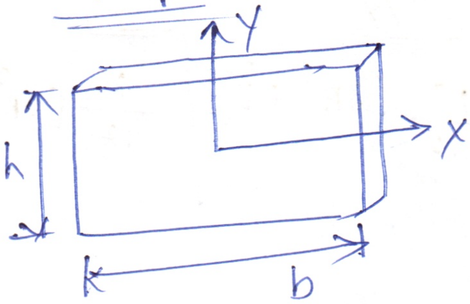
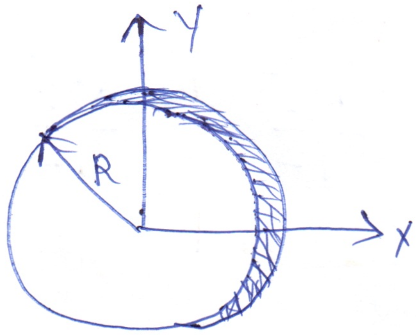
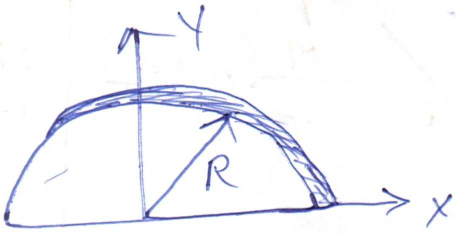
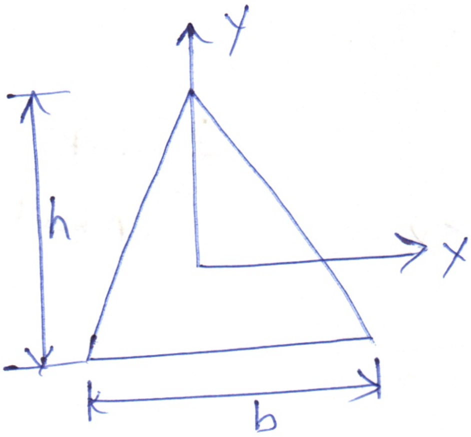
$$= \rho t \frac{bh^3}{12}$$

$m = \rho b h t$ $\quad \therefore I_{XX} = \frac{Mh^2}{12}$

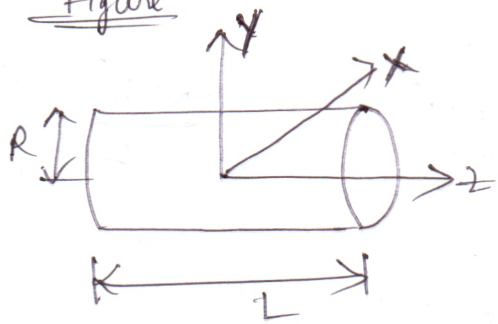
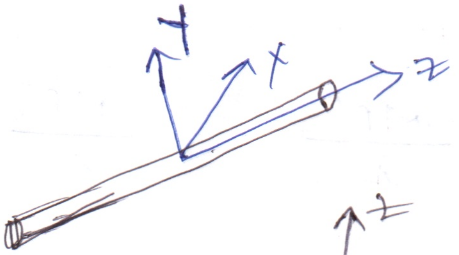
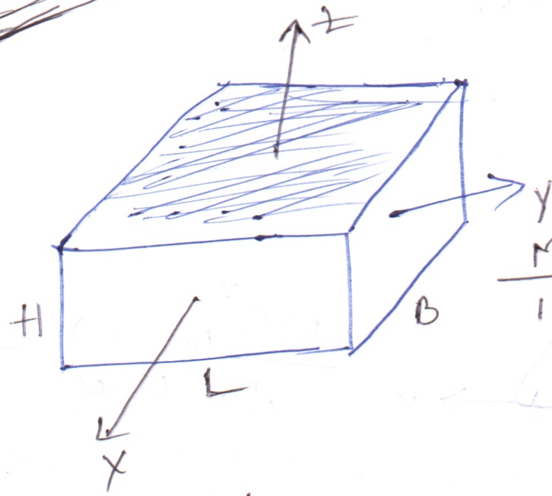
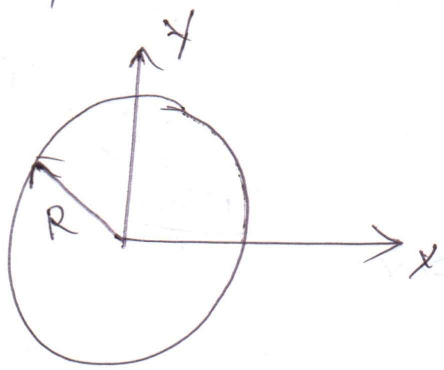
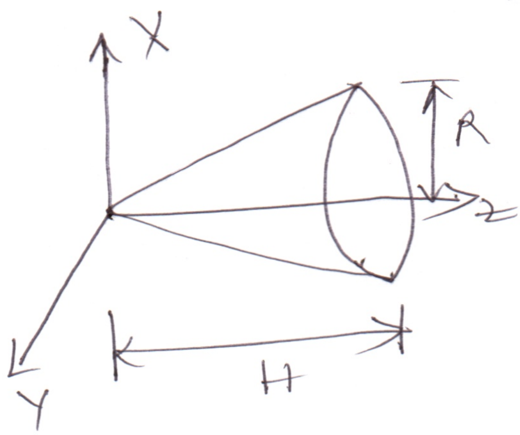
$$I_{YY} = \frac{mb^2}{12}$$

$$(I_{zz}) = \frac{M}{12} [b^2 + h^2]$$

Mass M.I of thin plates :

plate	shape	I_{xx}	I_{yy}
Rectangular		$\frac{Mh^2}{12}$	$\frac{Mb^2}{12}$
circular		$\frac{MR^4}{4}$	$\frac{MR^2}{4}$
Semi circular		$\frac{MR^2}{4}$ (About base)	$\frac{MR^4}{4}$
Triangular		$\frac{Mh^2}{18}$	$\frac{Mb^2}{24}$

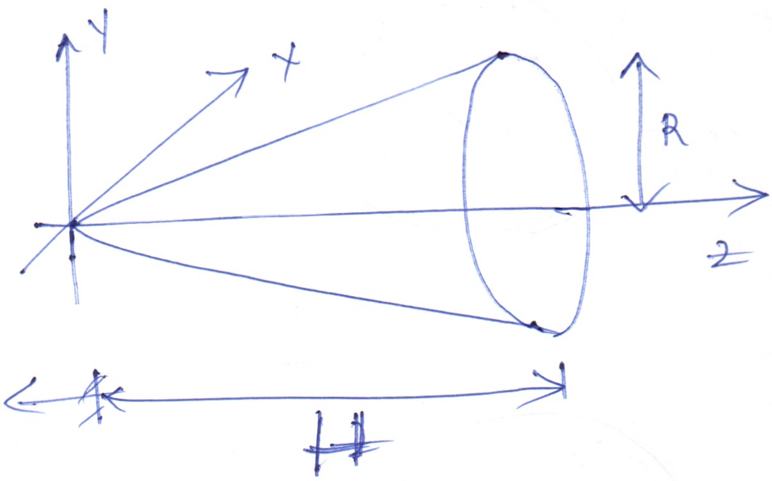
Mass Moment of inertia of solids :-

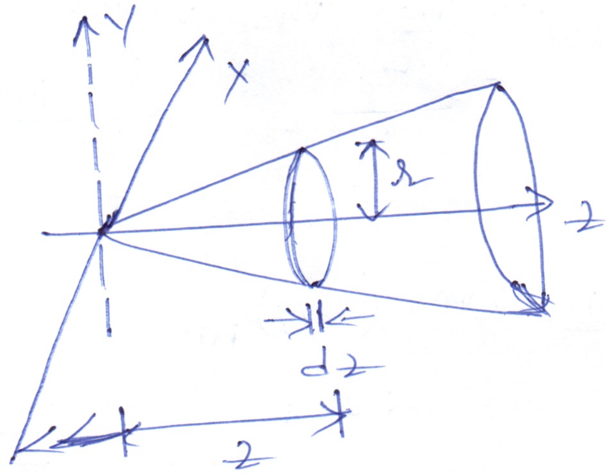
Shape	Figure	I_{xx}	I_{yy}	I_{zz}
Cylinder		$\frac{M}{12}[3R^2+L^2]$	$\frac{M}{12}[3R^2+L^2]$	$\frac{MR^2}{12}$
Slender rod		$\frac{ML^2}{12}$	$\frac{ML^2}{12}$	passing Axis
Prism		$\frac{M}{12}[L^2+H^2]$	$\frac{M}{12}[B^2+H^2]$	$\frac{M}{12}(L^2+B^2)$
Sphere		$\frac{2}{5}MR^2$ (About any diametric Axis)		
Cone :		$\frac{3}{5}M\left[\frac{R^2}{4}+H^2\right]$	$\frac{3}{5}M\left[\frac{R^2}{4}+H^2\right]$	$\frac{3}{10}[MR^2]$

Cone :-

Consider a cone of base radius 'R' & height "H" & mass density 'ρ', oriented w.r.to to the axes as shown in fig. Suppose we cut a circular radius 'r' & infinitesimal thickness dz at a distance 'z' from the origin. Then its mass is given as

$$\Rightarrow dm = \rho \pi r^2 dz$$





∴ its mass moment of inertia is given as

$$(dI_{zz})_{mass} = dm\, r^2/2 = \rho\left(\frac{\pi r^4}{2}\right) dz$$

on integration b/w limits

$$(I_{zz}) = \int_0^H \rho\left(\pi \frac{r^4}{2}\right) dz$$

similar triangles

$$r/R = z/H \Rightarrow \left[r = \frac{z}{H} \times R\right]$$

$$\therefore I_{zz} = \int_0^H \frac{\rho\pi}{2}\left[\frac{R^4}{H^4} z^4\right] dz$$

$$= \frac{\rho\pi}{2} \frac{R^4}{H^4} \int_0^H z^4\, dz$$

$$(I_{zz}) = \frac{\rho\pi}{2}\frac{R^4}{H^4}\frac{H^5}{5} = \frac{1}{10}\rho\pi R^4 H$$

we know that volume of the cone is

$$V = \frac{1}{3}\pi R^2 H$$

$$\therefore \quad M = \rho V = \frac{1}{3}\rho\pi R^2 H$$

Substituting these values $\left[M = \frac{1}{3}\pi r^2 \rho H\right)$

$$\Rightarrow I_{zz} = \frac{3}{10} M R^2$$

Calculation $(I_{xx})_{mass}$

$$dI'_{x'x'} = dm \frac{r^2}{4}$$

$$= \rho\pi r^2 dz\, r^2/4$$

$$(dI_{xx})_{mass} = \rho\left(\frac{\pi r^4}{4}\right)dz + \rho\pi r^2\, dz\, z^2$$