GENERAL PROCEDURE OF FEM/FEA :

(1) Descritization of domain into finite elements (known shaped)

→ Check for 1-D, 2-D, 3-D axial symmetric

→ Type of elements

→ Locations of nodes

(a) Point load existing

(b) Abrupt change of geometry

(c) Change of material

(d) Variation in load

→ No of elements (or) size of the element

coarse mesh - No of elements are less

fine mesh - No of elements are more

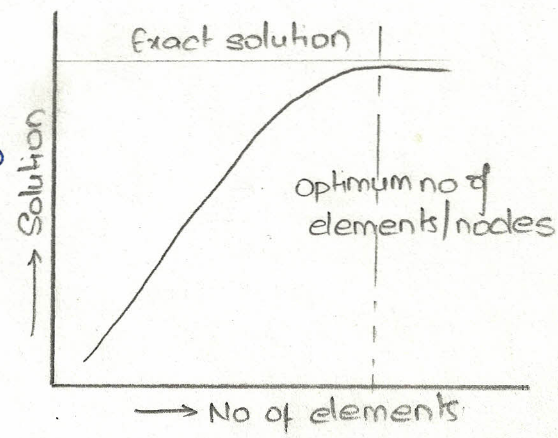
Errors incorporated

(a) Some material is ignored, (b) No. of Elements (c) Material Properties

(2) Selection of suitable interpolation function.

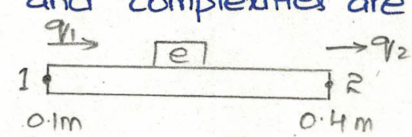
The variation of field variable with the element is assumed by a simple polynomial function is called interpolation function.

$I.f = f(x,y,z)$



LINEAR, QUADRATIC, CUBIC

→ Increasing the order of the polynomial the accuracy of the solution increases and complexities are more

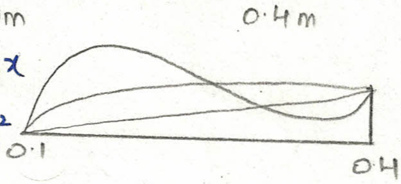


$f(x,y,z) = \underbrace{a_0 + a_1x + a_2y + a_3z}$ (linear)

(Quadratic) $\underbrace{+a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8yz + a_9zx}$

$+a_{10}x^3 + a_{11}x^2y + a_{12}xy^2 + a_{13}y^3 + a_{14}x^2z + a_{15}xz^2$

$+a_{16}z^3 + a_{17}y^2z + a_{18}yz^2$



(3) Estimation of elemental stiffness matrix and load vector.

$[K^e]_{n\times n}$ & $[P^e]_{n\times 1}$ n → no of nodes × Degree of Freedom (DOF)

Where $[K^e] = \int_V B^T D B \, dv$ B → Strain displacement relation matrix

D → Stress strain relation matrix

(4) Assemble the elements to obtain global stiffness matrix and global load vector (Through nodal connectivity)

$$[K] \ \& \ [P]$$

(5) Solve the equilibrium equation to obtain the nodal field variable vector.

$$[K]\cdot\{q\} = \{P\}$$

$[K] \longrightarrow$ Global stiffness matrix
$\{q\} \longrightarrow$ Global nodal field variable vector
$\{P\} \longrightarrow$ Global load vector

$$\{q\} = [K]^{-1}\{P\}$$

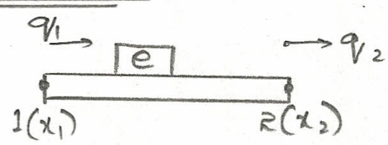
(6) Estimate the secondary variables, strains, stress, strain energy, reactions etc.

$$\epsilon = \frac{\Delta l}{l} = \frac{q_2 - q_1}{l}$$

$$\sigma = E\epsilon \ , \ U = \frac{1}{2}\sigma\epsilon V$$

Calculation of stiffness matrix for axial bar element

$\rightarrow$ Linear I.F of 1-D



$$q(x) = a_0 + a_1 x$$

at $x = x_1$, $q = q_1$

$x = x_2$, $q = q_2$

$$\rightarrow q_1 = a_0 + a_1 x_1 \quad \text{———} (1)$$

$$q_2 = a_0 + a_1 x_2 \quad \text{———} (2)$$

$$q_2 - q_1 = a_1(x_2 - x_1)$$

$$a_1 = \frac{q_2 - q_1}{x_2 - x_1}$$

$$q_1 = a_0 + \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1$$

$$a_0 = q_1 - \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1 = \frac{q_1 x_2 - \cancel{q_1 x_1} - q_2 x_1 + \cancel{q_1 x_1}}{x_2 - x_1}$$

$$\Rightarrow a_0 = \frac{q_1 x_2 - q_2 x_1}{x_2 - x_1}$$

$$\Rightarrow q(x) = \left[\frac{q_1 x_2 - q_2 x_1}{x_2 - x_1}\right] + \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x$$

$$= \frac{q_1 x_2 - q_2 x_1 + q_2 x - q_1 x}{x_2 - x_1}$$

$$q(x) = \left[\frac{x_2 - x}{x_2 - x_1}\right] q_1 + \left[\frac{x - x_1}{x_2 - x_1}\right] q_2$$

$$\epsilon = \frac{\Delta l}{l} = \frac{q_2 - q_1}{x_2 - x_1} = \frac{dq}{dx}$$

$$\epsilon = \frac{d}{dx}\left[\left(\frac{x_2 - x}{x_2 - x_1}\right) q_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) q_2\right]$$

$$\epsilon = \frac{-q_1}{x_2 - x_1} + \frac{q_2}{x_2 - x_1} = \left[-\frac{q_1}{l} + \frac{q_2}{l}\right]$$

$$\epsilon = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \Rightarrow \epsilon = Bq \; ; \; B = \left[\frac{-1}{l} , \frac{1}{l}\right]$$

$B \rightarrow$ Strain displacement matrix
$q \rightarrow$ Displacement matrix

$$[k^e] = \int_V B^T D B \, dv \qquad \boxed{\sigma = E\epsilon} \qquad \boxed{\sigma = D\epsilon}$$

$\boxed{\text{FOR } \underline{\text{1-D}} \quad E = D}$

$D \rightarrow$ Stress strain relation matrix

$$= \int_V \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}_{2\times1} [E]_{1\times1} \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}_{1\times2} dv$$

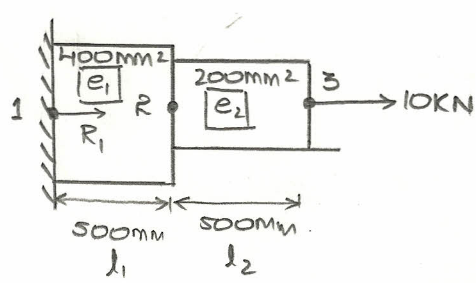
$$= \int_l E \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} A \, dl$$

$$[k^e] = \frac{AE}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_l dl = \frac{A^{[e]} E^{[e]}}{l^{[e]}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Problem

① Calculate the displacements, stresses, strains and reactions of a stepped bar as shown in fig $E = 2\times10^{11} N/m^2$

$$k^{[e]} = \frac{A^{[e]} E^{[e]}}{l^{[e]}} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \end{array}$$



$$k^{[1]} = \frac{A^{[1]} E^{[1]}}{l^{[1]}} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \end{array}$$

Global field nodal variable $\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$

$$\{P^{[1]}\} = \begin{array}{c c} \begin{Bmatrix} R_1 \\ 0 \end{Bmatrix} & \begin{array}{c} q_1 \\ q_2 \end{array} \end{array}$$

$$\{P^{[2]}\} = \begin{array}{c c} \begin{Bmatrix} 0 \\ 10\times10^3 \end{Bmatrix} & \begin{array}{c} q_2 \\ q_3 \end{array} \end{array}$$

$$[K^{[1]}] = \frac{400\times10^{-6}\times2\times10^{11}}{500\times10^{3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^8 \begin{array}{c c} \begin{array}{cc} q_1 & q_2 \end{array} & \\ \begin{bmatrix} 1.6 & -1.6 \\ -1.6 & 1.6 \end{bmatrix} & \begin{array}{c} q_1 \\ q_2 \end{array} \end{array}$$

$$[K^{[2]}] = \frac{200\times10^{-6}\times2\times10^{11}}{500\times10^{-3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^8 \begin{array}{c c} \begin{array}{cc} q_2 & q_3 \end{array} & \\ \begin{bmatrix} 0.8 & -0.8 \\ -0.8 & 0.8 \end{bmatrix} & \begin{array}{c} q_2 \\ q_3 \end{array} \end{array}$$

$$[K] = 10^8 \begin{array}{c c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} & \\ \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} & \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \end{array}$$

$$[K] = 10^8 \begin{bmatrix} 1.6 & 1.6 & 0 \\ -1.6 & \underset{=2.4}{(1.6+0.8)} & -0.8 \\ 0 & -0.8 & 0.8 \end{bmatrix}, \quad \text{Global } [P] = \begin{bmatrix} R_1 \\ 0 \\ 10^4 \end{bmatrix}$$

$$[K][q] = \{P\}$$

$$\Rightarrow 10^8 \begin{bmatrix} \cancel{1.6} & \cancel{-1.6} & \cancel{0} \\ \cancel{-1.6} & \boxed{2.4} & \boxed{-0.8} \\ \cancel{0} & \boxed{-0.8} & \boxed{0.8} \end{bmatrix} \begin{bmatrix} q_{1(0)} \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 10^4 \end{bmatrix}$$

$q_1 = 0$ because node one is fixed

$$10^8 \begin{bmatrix} 2.4 & -0.8 \\ -0.8 & 0.8 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

$$10^8 (2.4 q_2 - 0.8 q_3) = 0 \quad \text{——— (1)}$$

$$10^8 (-0.8 q_2 + 0.8 q_3) = 10^4 \quad \text{——— (2)}$$

$$10^8 (1.6 q_2) = 10^4$$

$$q_2 = \frac{10^4}{10^8} \times \frac{1}{1.6} = 0.0000625 \text{ m}$$

$$q_3 = \frac{10^8 \times 2.4 (0.0000625)}{0.8} = 0.0001875 \text{ m}$$

$$\varepsilon^{[1]} = \frac{q_2 - q_1}{l^{[1]}} = \frac{0.0000625}{l^{[1]}} = 1.25\times10^{-5}$$

$$E^{[2]} = \frac{q_3 - q_2}{l^{[2]}} = 2.5\times10^{-5}$$

$$\sigma^{[1]} = E^{[1]}\epsilon^{[1]} = 2\times10^{5}\times1.25\times10^{-5} = 2.5\ N/mm^2$$

$$\sigma^{[2]} = E^{[2]}\epsilon^{[2]} = 2\times10^{5}\times2.5\times10^{-5} = 5\ N/mm^2$$

→ To Calculate the reactions

$$\Rightarrow 10^8 \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 2.4 & -0.8 \\ 0 & -0.8 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 10^4 \end{bmatrix}$$

$$\Rightarrow R_1 = 10^8\left(1.6\times0 - 1.6(q_1) + 0\times q_2\right)$$

$$R_1 = -10^4 N \quad = 10\,KN$$

Strain energy

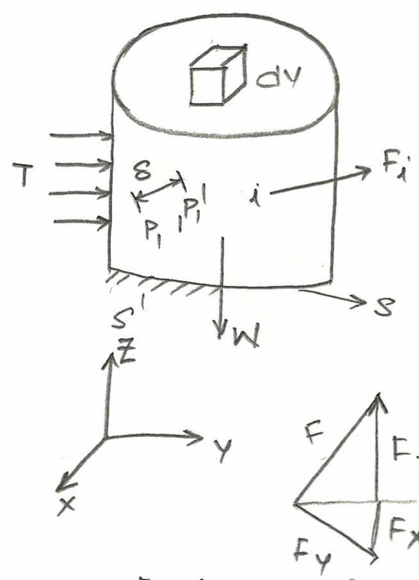
$$U_1^{[1]} = \frac{1}{2}\sigma_1\,\epsilon' V' = \frac{1}{2}\times2.5\times10^{6}\times1.25\times10^{-5}\times400\times10^{-6}\times0.5 = 3125\ N\text{-}m \text{ or } J$$

$$U^{[2]} = \frac{1}{2}\sigma^2\epsilon^2 V^2 =$$

STRESS & EQUILIBRIUM

Stress strain relations:

Consider a three dimensional body of
volume − V
Surface − S
constrained position of surface − S'
Point load applied at the point − F_i
Traction Force − T (i.e surface)
displacement − δ
P_1 → Location of point before loading
P_1' → " " " after "
W → Body force (i.e weight of body)
dv → Elementary volume



$F = (F_x\ F_y\ F_z)$

→ For analysis the loads or stresses applied from many directions are resolved into the directions of the universal system of axes as x, y and z. When a body which is subjected to different types of ~~stresses~~ forces is to be kept in stable condi-tion (i.e equilibrium state). The algebraic sum of the forces in x, y, z directions must be zero.

i.e $\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$. The rule is applied for stresses also i.e the stresses $\Sigma \sigma_x = 0$, $\Sigma \sigma_y = 0$ and $\Sigma \sigma_z = 0$

→ For concept analysis. the points, forces, displacements etc must be specified in terms of global axes system i.e (x, y, z)

That is point 'P' must be specified as $P(x, y, z)$

Force 'F' must be specified as $F = (F_x, F_y, F_z)$

Distributed force per unit volume $f = (f_x, f_y, f_z)$

Body force like weight	$W = (W_x, W_y, W_z)$
Surface force	$T = (T_x, T_y, T_z)$
Displacement	$\delta = (u, v, w)$

Hooke's Law $\quad \sigma \propto \epsilon$ (Stress ∝ Strain)

$\sigma = E\epsilon \qquad E \longrightarrow$ Young's modulus (or) rigidity modulus

3 Normal stresses $= \sigma_x, \sigma_y, \sigma_z$

3 Shear stresses $= \tau_{xy}, \tau_{yz}, \tau_{zx}$

3 Normal strains $= \epsilon_x, \epsilon_y, \epsilon_z$

3 Shear strains $= \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

Generalized Hooke's Law and equations

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu$$

$\mu \longrightarrow$ Poisson's ratio

$$\epsilon_y = -\frac{\sigma_x \mu}{E} + \frac{\sigma_y}{E} - \frac{\sigma_z \mu}{E}$$

$$\epsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$G \longrightarrow$ Rigidity modulus

$$G = \frac{E}{2(1+\mu)}$$

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Boundary conditions:

$u = 0$ on s'

Stress - Strain Relationship matrix for 3-D

$$\left.\begin{aligned} E\epsilon_x &= \sigma_x - \mu\sigma_y - \mu\sigma_z \\ E\epsilon_y &= -\mu\sigma_x + \sigma_y - \mu\sigma_z \\ E\epsilon_z &= -\mu\sigma_x - \mu\sigma_y + \sigma_z \end{aligned}\right\} \text{Solve for } \sigma_x, \sigma_y, \sigma_z$$

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)}\left[\epsilon_x(1-\mu) + \mu\epsilon_y + \mu\epsilon_z\right]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)}\left[\epsilon_x\mu + \epsilon_y(1-\mu) + \mu\epsilon_z\right]$$

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)}\left[\epsilon_x\mu + \mu\epsilon_y + (1-\mu)\epsilon_z\right]$$

σ_x σ_y σ_z τ_{xy}

Shear stress and shear strain relationships

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad G = \frac{E}{2(1+\mu)}$$

$$\gamma_{xy} = \frac{\tau_{xy}\times 2(1+\mu)}{E}$$

$$\therefore \tau_{xy} = \frac{E}{(1+\mu)2}\times\gamma_{xy} = \frac{E}{(1+\mu)(1-2\mu)}\times\left(\frac{1-2\mu}{2}\right)\gamma_{xy}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}_{6\times1} = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix}1-\mu & \mu & \mu & 0 & 0 & 0\\ \mu & 1-\mu & \mu & 0 & 0 & 0\\ \mu & \mu & 1-\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2}\end{bmatrix}_{6\times6}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}_{6\times1}$$

The above eq^n can also be expressed as

$$\{\sigma\}_{6\times1} = [D]_{6\times6}[\epsilon]_{6\times1}$$

For 2-D system

For 2-D system the displacement problems are modelled as plane stress and plane strain problems.

Plane stress: $\sigma_z = 0$; τ_{yz} & $\tau_{zx} = 0$; $\gamma_{yz} = \gamma_{zx} = 0$

But ϵ_z will be produced by the stress σ_y & σ_x

$$\therefore \epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$$

$$\epsilon_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\epsilon_z = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E} \tau_{xy}$$

To find σ_x, σ_y & τ_{xy}

$$\sigma_x = \frac{E}{1-\mu^2} (\epsilon_x + \mu \epsilon_y)$$

$$\sigma_y = \frac{E}{1-\mu^2} (\mu \epsilon_x + \epsilon_y)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy} = \frac{E}{(1-\mu^2)} \left[\frac{1-\mu}{2}\right] \gamma_{xy}$$

Writing the above eq^n in matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\sigma = [D]\{\epsilon\}$$

Plane Strain:

Long body of uniform c/s

$\epsilon_z = 0, \ \gamma_{yz} = \gamma_{zx} = 0, \ \sigma_z \neq 0$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z = 0 \\ \gamma_{xy} \\ \gamma_{yz} = 0 \\ \gamma_{zx} = 0 \end{bmatrix}$$

Since $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ Neglect $3^{rd}, 5^{th}$ & 6^{th} rows & columns

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} [\mu \epsilon_x + \mu \epsilon_y]$$

1-D

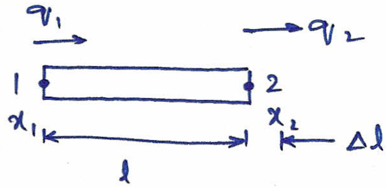
$$[D]_{1 \times 1} = E$$

$$\sigma = E \epsilon \qquad E = D \qquad \epsilon = Bq$$

Stress Displacement Relations

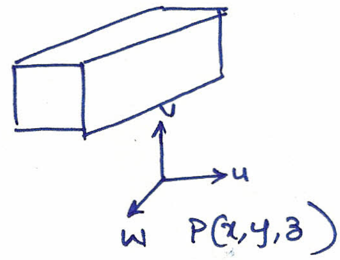
1D

$$\epsilon = \frac{\Delta l}{l} = \frac{q_2 - q_1}{x_2 - x_1} = \frac{dq}{dx}$$



3D

$$[\epsilon] = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{bmatrix}$$



2D

$$[\epsilon] = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

1-D $\{\epsilon\} = \{\epsilon_x\} = \left\{\frac{\partial u}{\partial x}\right\} = \frac{dq}{dx}$

Minimum Potential Energy Principle (or) Virtual Energy Principle for Formulation of Stiffness Matrix

Total Potential energy (Π) = Sum of internal strain energy + External work done

$\Pi = U + W$ W will be -ve when the work is done on the system

$\Pi = U - W$ —(1) U → Strain energy $\boxed{W = Pq}$ (force × displacement)

→ The principle of minimum potential energy states that when the total potential energy Π is min (as in stable equilibrium) its first domain is zero

i.e if Π min then $\frac{d\Pi}{dx} = 0$, x → displacement

Loads P → displacement → strains → stress → strain energy

Strain Energy: $U = \frac{1}{2} \times$ stress × strain × volume

$$U = \int_V \frac{1}{2} \sigma^T \epsilon \, dV = \int_V \frac{1}{2} (D\epsilon)^T \epsilon \, dV = \int_V \frac{1}{2} (DBV)^T Bq \, dV$$

$$\boxed{U = \frac{1}{2}\int_V q^T B^T D^T B q \, dv} \text{ ——— (2)}$$

→ Let x may be the displacement and k may be stiffness of a member

$$\text{Strain Energy} = \frac{1}{2} k x^2$$

$$U = \frac{1}{2} [K]_{n\times n} \{q\}^2_{n\times 1}$$

$$U = \frac{1}{2} q^T_{1\times n} K_{n\times n} q_{n\times 1} \text{ ——— (3)}$$

Equalise (2) & (3)

$$\frac{1}{2}\int_V q^T B^T D^T B q \, dv = \frac{1}{2} q^T K q \qquad [D^T = D]$$

$$\frac{1}{2} q^T \left[\int_V B^T D B \, dv\right] q = \frac{1}{2} q^T K_{n\times n} q$$

$$\boxed{[K^{[e]}] = \int_V B^T D B \, dv} \text{ ——— (4)}$$

$$\Pi = U - W$$

$$= \frac{1}{2} q^T K q - q^T P = \frac{1}{2} q^2 K - qP$$

According to minimum potential energy principle under equilibrium conditions.

$$\boxed{\frac{d(q^2)}{dq} = 2q}$$

$$\frac{\partial \Pi}{\partial q} = 0 \qquad \cancel{2} \times \frac{1}{\cancel{2}} qK - qP = 0$$

$$Kq - P = 0$$

$$Kq = P$$

$[K]_{n\times n} \{q\}_{n\times 1} = \{P\}$ → equilibrium eq^n (or) finite element eq^n

$$\{q\} = [K^{-1}]\{P\}$$

$$\epsilon = [B]\{q\}$$

$$\{\sigma\} = [D]\{\epsilon\}$$

NOTE : Transpose is taken for multiplication purpose
There may not be any significant ~~use~~ otherwise
Remove transpose when not required

Example: $[B]$ is 1×2 matrix in case of axial bar element

$$[B] = \begin{bmatrix} -1/l & 1/l \end{bmatrix}, \quad [B^T] = \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}$$

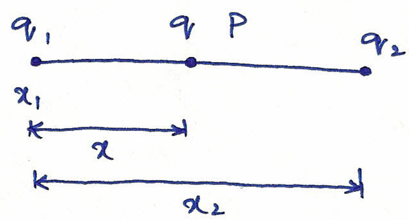
$[B]_{1\times 2} [B^T]_{2\times 1}$ → can be multiplied

$[B]_{1\times 2} [B]_{1\times 2}$ → can't be multiplied

Shape Functions :

Shape functions are the mathematical expressions to define the geometry or shape of the finite element. By using the shape functions, the variation of field variable like displacement, temperature with in the finite element can be judged. These functions are mostly associated with the nodal values of the element and are specified by co-ordinate values.

Expressing field variable is quadratic



$$q(x) = a_0 + a_1 x$$

$$q(x) = N_1 q_1 + N_2 q_2$$

The value of field variable as P

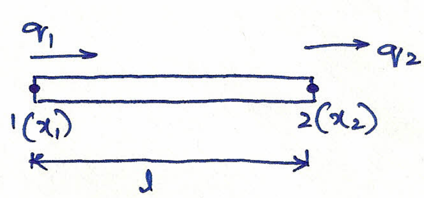
q_1 & q_2 are nodal values

N_1 & N_2 are shape functions

→ In one dimensional elements the displacements occuring at the two ends (i.e primary modes) can be determined using suitable finite elements equations. At the same the displacement in other locations of the elements (i.e other than primary nodes) can be calculated with the help of some other functions called shape functions which are functions of co-ordinates. That is once the shape functions are defined the linear displacement of any location with in the element can be specified in terms of primary nodal displacements.

Calculation of shape function :

Axial Bar element :



$$q(x) = a_0 + a_1 x$$

at $x = x_1$; $q = q_1$

at $x = x_2$; $q = q_2$

$$q_1 = a_0 + a_1 x_1 \text{ ——(2)}$$

$$q_2 = a_0 + a_1 x_2 \text{ ——(3)}$$

$$q_2 - q_1 = a_1 (x_2 - x_1)$$

$$a_1 = \frac{q_2 - q_1}{x_2 - x_1}$$

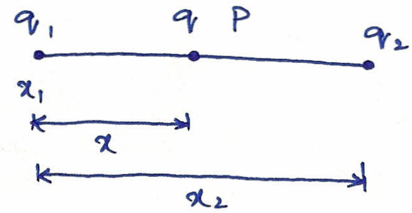
$$q_1 = a_0 + \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1$$

$$a_0 = q_1 - \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1$$

Shape Functions:

Shape functions are the mathematical expressions to define the geometry or shape of the finite element. By using the shape functions, the variation of field variable like displacement, temperature with in the finite element can be judged. These functions are mostly associated with the model values of the element and are specified by co-ordinate values.

Expressing field variable is quadratic



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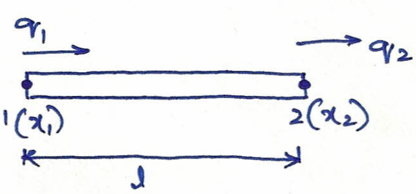
Calculation of shape function:

Axial Bar element:



$q(x) = a_0 + a_1 x$

at $x = x_1$; $q = q_1$

at $x = x_2$; $q = q_2$

$$q_1 = a_0 + a_1 x_1 \quad \text{——(2)}$$

$$q_2 = a_0 + a_1 x_2 \quad \text{——(3)}$$

$$q_2 - q_1 = a_1 (x_2 - x_1)$$

$$a_1 = \frac{q_2 - q_1}{x_2 - x_1}$$

$$q_1 = a_0 + \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1$$

$$a_0 = q_1 - \left[\frac{q_2 - q_1}{x_2 - x_1}\right] x_1$$

$$a_0 = \frac{q_1x_2 - \cancel{q_1x_1} - q_2x_1 + \cancel{q_1x_1}}{x_2 - x_1} = \frac{q_1x_2 - q_2x_1}{x_2 - x_1}$$

$$q(x) = \frac{q_1x_2 - q_2x_1}{x_2 - x_1} + \left[\frac{q_2 - q_1}{x_2 - x_1}\right]x$$

$$q(x) = \frac{q_1x_2 - q_2x_1 + q_2x - q_1x}{x_2 - x_1} = \left[\frac{x_2 - x}{x_2 - x_1}\right]q_1 + \left[\frac{x - x_1}{x_2 - x_1}\right]$$

$$= \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1}\right]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$= [N_1 \quad N_2]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$q(x) = [N \quad q] \longrightarrow$ shape function matrix

Where $N_1 = \frac{x_2 - x}{x_2 - x_1}$, $N_2 = \frac{x - x_1}{x_2 - x_1}$; N_1 & N_2 are shape functions

$$q(x) = N_1q_1 + N_2q_2$$

Characteristics of shape functions :

1. No of shape functions in the elements are equal to no of nodes.
2. The sum of shape functions are equal to one within the elements $\Sigma N_i = 1$ at any location with in the elements

   where i = Nodal number

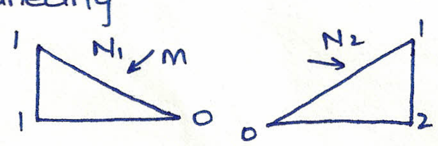
   i.e $N_1 + N_2 = 1$, $\frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$
3. The value of shape function $N_i = 1$ at Node i, and zero at all other nodes.

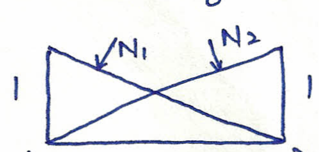
   At $x = x_1$ as node 1

   the value of $N_1 = 1$ and $N_2 = 0$ $\frac{\cancel{x_1 - x_1}^{\,0}}{x_2 - x_1} = 0$, $N_1 = \frac{x_2 - \cancel{x_1}}{x_2 - \cancel{x_1}} = 1$

   as $x = x_2$ at node 2

   $N_2 = \frac{x - x_1}{x_2 - x_1} = 1$
4. The value of 'i' is varying from 1 to 0 from node 1 to other nodes linearly





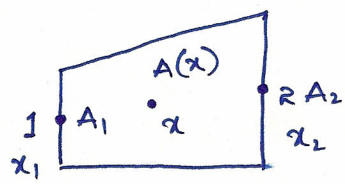
~~Remaining two points leave some space~~

(5) The space functions of the elements are functions of space co-ordin-ates only (x, y & z)

Isoparametric Representation :

If any of the parameters varies as per the order of poly of I.F its value with in the elements can be represented by using the shape functions called iso parametric representation.

$$\boxed{A(x) = N_1 A_1 + N_2 A_2}$$

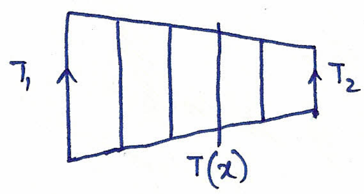


$$A(x) = A_1 + \left[\frac{A_2 - A_1}{x_2 - x_1}\right](x - x_1)$$

$$= \frac{A_1 x_2 - \cancel{A_1 x_1} + A_2 x - A_1 x - A_2 x_1 + \cancel{A_1 x_1}}{x_2 - x_1}$$

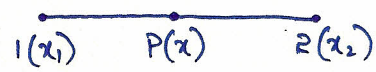
$$= A_1 \frac{(x_2 - x)}{x_2 - x_1} + A_2 \left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$\boxed{A(x) = N_1 A_1 + N_2 A_2}$$



$$T(x) = T_1 N_1 + T_2 N_2$$

$$P(x) = N_1 x_1 + N_2 x_2$$



~~Intersection~~ Integration of shape functions

For 1-D

$$\int_{l(e)} N_1^p N_2^q \, dx = \frac{p!\,q!}{(p+q+1)!} l^e$$

For 2-D

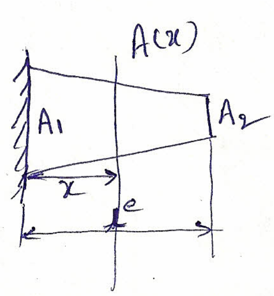
$$\int_{A(e)} N_1^p N_2^q N_3^r \, dA = \frac{p!\,q!\,r!}{(p+q+r+2)!} 2A^e$$

For 3-D

$$\int_{V(e)} N_1^p N_2^q N_3^r N_4^s \, dV = \frac{p!\,q!\,r!\,s!}{(p+q+r+s+3)!} 6V^e$$

For Example :

$$\int_{l^e} N_1^2 \, dx = \begin{matrix} p = 2 \\ q = 0 \end{matrix} \left| \frac{2! \times 0!}{(2+0+1)!} l^e = \frac{2}{6} l = \frac{l^3}{3} \right.$$



REMAINING IS CONTINUED IN Pg 15...

Axial B~~eam~~ar Element

Calculation of elemental stiffness matrix by using shape function.

$$u_1 = N_1 q_1 + N_2 q_2$$

$$\epsilon = \frac{du}{dx} = \frac{d}{dx}\left(N_1 q_1 + N_2 q_2\right) \qquad N_1 = \frac{x_2 - x}{x_2 - x_1} \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$= \frac{dN_1}{dx} q_1 + \frac{dN_2}{dx} q_2$$

$$\epsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = B \cdot q$$

$$\Rightarrow \frac{dN_1}{dx} \quad \frac{dN_2}{dx} = \begin{bmatrix} \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix} = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$K^{[e]} = \int_V B^T D B \, dv$$

$$= \int_V \frac{1}{l^e} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} E^e \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} A^e dx$$

$$= \frac{1}{l^{e2}} A^{[e]} E^{[e]} \int_{l^e} dx \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{l^{\cancel{2}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cancel{l}$$

$$= \frac{A^e E^e}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{[e]}] = \int_V B^T D B \underbrace{dv}_{A\,dx} = \int_V B^T D B \underbrace{(N_1 A_1 + N_2 A_2)}_{A(x)} dx$$

REMAINING PART OF
Pg 13

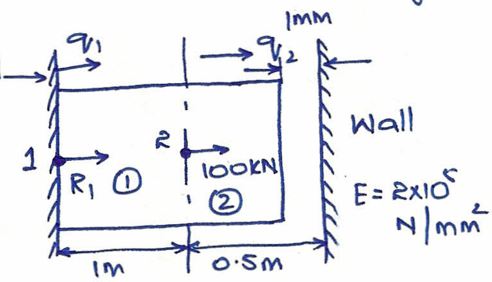
$$= \frac{E^{[e]}}{l^{[e]}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[\frac{A_1 l^e}{2} + \frac{A_2 l^e}{2} \right]$$

$$\boxed{\int_l N_1\,dx = \frac{l^e}{2} \quad P = 1}$$

$$= \frac{[A_1 + A_2]}{2 l^e} E^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

---

(1) Calculate displacements, strains, stresses and reactions shown in fig



$$K^{[e]} = \frac{A^{[e]} E^{[e]}}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{[1]} = \frac{400\times10^{-6} \times 2\times1.5\times10^{6}}{1} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \end{array}$$

$$K^{[1]} = 8\times10^{7} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \end{array}$$

$$K^{[2]} = 8\times10^{7} \begin{array}{c} \begin{array}{cc} q_2 & q_3 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} q_2 \\ q_3 \end{array} = \frac{400\times10^{-6}\times2\times10^{5}\times10^{-6}}{0.5}$$

$$[P^1] \text{(load vector)} = \left\{ \begin{array}{c} R_1 \\ 100\times10^3 \end{array} \right\} \begin{array}{c} q_1 \\ q_2 \end{array}$$

$$[K] = 8\times10^{7} \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \qquad P^2 = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

$$= 8\times10^{7} \begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} \cancel{q_1} \\ q_2 \\ q_3 \end{bmatrix} \begin{bmatrix} \cancel{R_1} \\ 100\times10^3 \\ 0 \end{bmatrix}$$

$$= 8\times10^{7} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 100\times10^3 \\ 0 \end{bmatrix}$$

$q_2 = q_3$ as no load is applied on element ②

$$\Rightarrow 8\times10^{7}(3q_2 - 2q_3) = 100\times10^{3}$$

$$8\times10^{7}(-2q_2 + 2q_3) = 0$$

$$q_2 = q_3$$

$$q_2 = \frac{100\times10^{3}}{8\times10^{7}} = 1.25\times10^{-3}\,\text{m} = 1.25\,\text{mm}$$

$q_3 > 1\,mm$

So reaction occurs at q_3

$$\Rightarrow 8\times10^7 \begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 3 & 2 \\ \cancel{0} & -2 & 2 \end{bmatrix} \begin{bmatrix} \cancel{q_1=0} \\ q_2 \\ q_3 = 1\times10^{-3} \end{bmatrix} = \begin{bmatrix} \cancel{R_1} \\ 100\times10^3 \\ R_3 \end{bmatrix}$$

$$\Rightarrow 8\times10^7 \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} q_2 \\ 1\times10^{-3} \end{bmatrix} = \begin{bmatrix} 100\times10^3 \\ R_3 \end{bmatrix}$$

$$3q_2 - 2\times10^{-3} = \frac{100\times10^3}{8\times10^7}$$

$$q_2 = 1.08\times10^{-3}\,m = 1.08\,mm$$

$$R_3 = 8\times10^7\left(-2\times1.08\times10^{-3} + 2\times1\times10^{-3}\right)$$

$$R_3 = -12.8\,KN$$

$$\Rightarrow 8\times10^7\left(1\times10 - 1\times1.08\times10^{-3} + 0\times1\times10^{-3}\right) = R_1$$

$$R_1 = -86.4\,KN$$

$$\epsilon^{[1]} = \frac{q_2 - q_1}{l^{[1]}} = \frac{1.08 - 0}{1\times10^3} = 1.08\times10^{-3}$$

$$\epsilon^{[2]} = \frac{q_3 - q_2}{l^{[2]}} = \frac{1 - 1.08}{0.5\times10^3} = -1.6\times10^{-4}$$

$$\sigma^{[1]} = E^{[1]}\epsilon^{[1]} = 2\times10^5 \times 1.08\times10^{-3} = 216\,N/mm^2$$

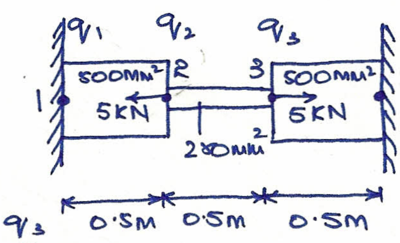
$$\sigma^{[2]} = E^{[2]}\epsilon^{[2]} = 2\times10^5 \times -1.6\times10^{-4} = -32\,N/mm^2$$

(2) Calculate displacement vector and stress for the following fig

$E = 2\times10^5\,N/m^2$

$$K^{[1]} = \frac{500\times10^{-6}\times2\times10^5\times10^6}{0.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 2\times10^8 \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \end{array}$$



$$K^{[2]} = \frac{250\times10^{-6}\times2\times10^5\times10^6}{0.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1\times10^8 \begin{array}{c} \begin{array}{cc} q_2 & q_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_2 \\ q_3 \end{array}$$

$$K^{[3]} = 2\times10^8 \begin{array}{c} \begin{array}{cc} q_3 & q_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} q_3 \\ q_4 \end{array}$$

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix}$$

$$[k] = 2\times10^{8}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & \underset{=1.5}{(1+0.5)} & -0.5 & 0 \\ 0 & -0.5 & \underset{1.5}{(0.5+1)} & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} q_1=0 \\ q_2 \\ q_3 \\ q_4=0 \end{bmatrix} = \begin{bmatrix} R_1 \\ -5\times10^{3} \\ 5\times10^{3} \\ R_2 \end{bmatrix}$$

$$\Rightarrow \quad 2\times10^{8}\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -5\times10^{3} \\ 5\times10^{3} \end{bmatrix}$$

$$2\times10^{8}(1.5q_2 - 0.5q_3) = -5\times10^{3}$$

$$2\times10^{8}(-0.5q_2 + 1.5q_3) = 5\times10^{3} \quad \times 3$$

$$2\times10^{8}(4q_3) = 10\times10^{3}$$

$$q_3 = 1.25\times10^{-5}\,m$$

$$2\times10^{8}(1.5q_2 - 0.5\times1.25\times10^{-5}) = -5\times10^{3}$$

$$q_2 = -1.25\times10^{-5}$$

$$\Rightarrow (1\times0 - 1\times -1.25\times10^{-5} + 0 + 0)\,2\times10^{8} \Rightarrow \boxed{R_1 = 1.25\,KN}$$

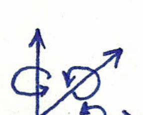
$$\Rightarrow R_2 = (0\times0\times q_2 - 1\times1.25\times10^{-5} + 0)\,2\times10^{5} \Rightarrow \boxed{R_2 = -2.5\,KN}$$

Stiffness Matrix [S.M] $k^{[e]}$ - Its Characteristics

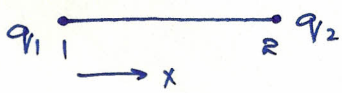
1. The SM of any element is a function of geometric co-ordinates and material properties.
2. The SM is a square matrix $(n\times n)$

   Where n = No of nodes × Degree of freedom of each node

Each node having max 6 degrees of freedom (Depends on loads applied.



→ Axial Bar Element



one degree of freedom
only one direction

$n = 2$ nodes $\times$ 1 DOF $= 2$, $\boxed{2\times 2}$

3. It is always a symmetric matrix

$$\left[K^{[e]}\right]^T = \left[K^{[e]}\right]$$

4. The sum of the numerical values at any row and column will be zero.

5. The determinants of the SM will always be zero.

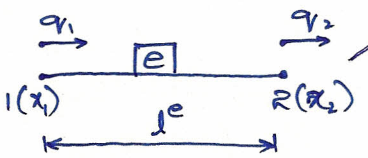
Types of Elements:

(1) 1-D Elements or (Linear elements)

its c/s area is uniform

(i) Axial Bar element

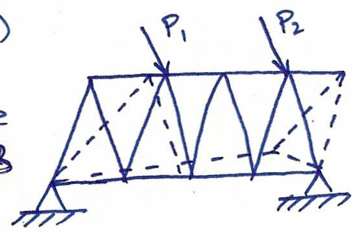
Load acting along axis and displacement exists only along the axis.



Size of SM = 2 Nodes × 1 DOF = 2
$\boxed{2\times 2}$

(ii) Truss element

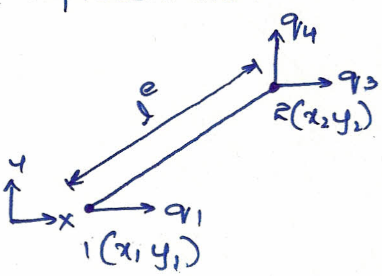
Plane Truss



Supporting structure for Bridges : Relation is not possible

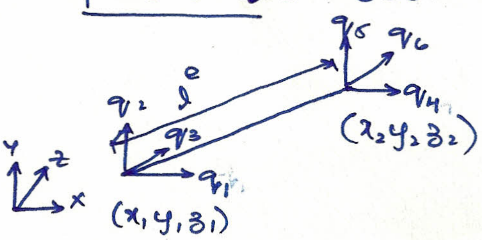
→ 2 DOF in x and y direction

Representation



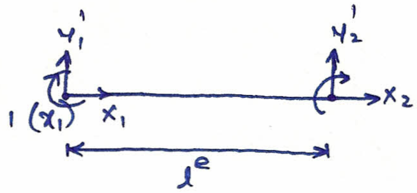
Size of SM = 2 Nodes × 2 DOF = 4
$\boxed{4\times 4}$

Space Stress 3 DOF (x y & z)



Size of SM = 2 Nodes × 3 DOF = 6
$\boxed{6\times 6}$

Frame Elements : Combination of bar and beam



Size = 2 Nodes × 3 DOF = 6

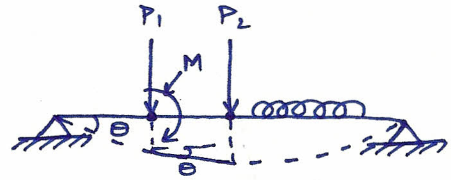
$\boxed{6 \times 6}$

Beam Elements : Load acting ⊥r or transverse direction

→ Assuming displacements x-direction negative. Assumption made while analysing the beam.

Representation of beam element

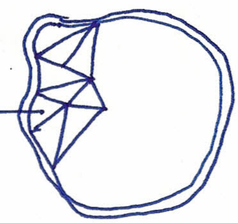
SM size = 2 × 2 = 4
= $\boxed{4 \times 4}$



DOF would Y and rotation about 'z' node will rotate

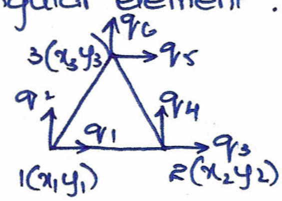
2-D Elements (or) Area Elements :

Thickness of elements is uniform and considering certain area we treat in 2-D element



Triangular element

(1) Triangular element :



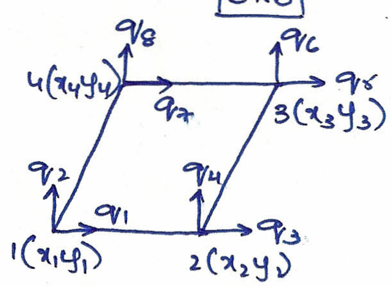
SM size = 3 Nodes × DOF 2 = 6

$\boxed{6 \times 6}$

(2) Quadratic element :

Size of SM = 4 Nodes × 2 DOF = 8

$\boxed{8 \times 8}$

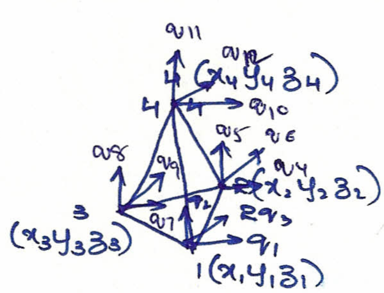


3-D Elements (or) Volume Elements :

Tetrahedral Element :

Size of SM = 3 DOF × 4 Nodes = 12

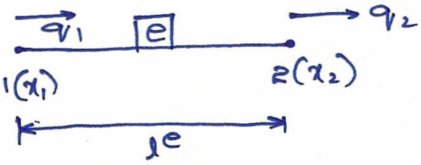
$\boxed{12 \times 12}$



Classification of Elements

Basic Elements

1. If the order of the polynomial of IF is linear then we can call it basic element.

2. 

$q(x) = a_0 + a_1 x$

at $x = x_1$, $q = q_1$ } Boundary
at $x = x_2$, $q = q_2$ } conditions

3. No of nodes in the element will be min.

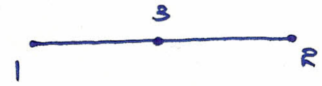
4. Size of SM is min 2x2

5. Less no of complications are involved.

6. Accuracy is less.

Higher Order Element

1. Polynomial I.F is quadratic call it is higher order.

2. 

$q(x) = a_0 + a_1 x + a_2 x^2$

Element axial bar only

3. More no of nodes are to be added according to the order of the polynomial.

4. Size of SM is large

5. More no of complications

6. Accuracy is high.

Applications of FEM :

(1) Civil Engineering point of view:

(a) For structural engg :- Design of buildings, towers, bridges, dams, trusses, frames, columns etc.

(b) Geotechnical engg :- To evaluate the strength of soil foundations etc.

(c) Water resources engg :- Design of canals, dams, velocity profiles etc.

(2) Mechanical Engineering

(a) Manufacturing stream :- Design of machine tool components, work pieces, M.T beds, cutting tools etc.

(b) Production stream :- To calculate residual stress during casting, bending, metal forming, heat treatment etc.

(c) Thermal Engineering :- Estimation of temp profiles, heat fluxes, heat losses etc. Design of I.C engines, G.T, S.T, H.E etc.

(3) Fluid Mechanics :

To evaluate velocity and pressure profiles.

(4) Aerospace Engg :

Design of air craft stresses, wings, airofoil m/c components etc.

(5) Ocean Engg:
Design of ships and submarine structure, ship body design, components of engine.

(6) Bio medical Engg:
To evaluate the strength of bones, pressure in blood tissues, for nerve system, eye balls etc.

(7) Nuclear Engg:
Analysis of nuclear pressure vessels ~~and~~

(8) Electrical Machinery & Electromagnetics:
Analysis of synchronous and induction machines eddy currents and core losses in electrical machines.

Advantages of FEM:

Using FEM we are able to

(1) Model irregular shaped bodies quite easily.
(2) Handle general load conditions with out difficulty.
(3) Model bodies composed of several different material because the element equations are evaluated individually.
(4) Handle unlimited numbers and kinds of boundary conditions.
(5) Include dynamic effects.
(6) After the finite elements model easily and cheaply.

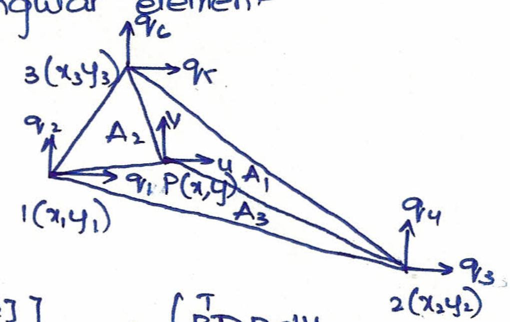
Disadvantages of FEM:

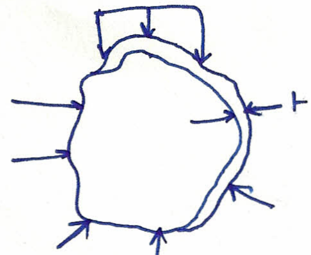
(1) The finite element method is a time consuming process, in requires longer time for solving.
(2) In order to analyse many no of smaller elements, due to human fatigue, we have to depend on computer package.
(3) The result obtained using FEM will be closer to exact solution only if the system is devided into large no of smaller elements.
(4) FEM cannot produce exact results as those of analytical methods.
(5) If we are not having sound background in mathematics, especially in matrix algebra, differentiation, integration, then solving problem using FEM is highly difficult.

2-D FE Analysis

Basic Elements → 3 Noded

Triangular element





Nodal displacement vector $\{q\} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$

$$\left[K^{[e]}\right]_{6\times6} = \int_V B^T D B \, dV$$

Displacement vector $\bar{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

$$\bar{u} = \begin{matrix} u = \\ v = \end{matrix}\begin{Bmatrix} a_0 + a_1x + a_2y \\ a_3 + a_4x + a_5y \end{Bmatrix} \begin{matrix} \text{——(1)} \\ \text{——(2)} \end{matrix}$$

Boundary conditions

At $x = x_1$ & $y = y_1$, $u = q_1$ ——(3)

$x = x_1$ & $y = y_1$, $v = q_2$ ——(4)

At $x = x_2$ & $y = y_2$ $u = q_3$ ——(5)

$v = q_4$ ——(6)

At $x = x_3$ & $y = y_3$ $u = q_5$ ——(7)

$v = q_6$ ——(8)

eq^{n}s (3)(5) & (7) in (1)

$$q_1 = a_0 + a_1x_1 + a_2y$$

$$q_3 = a_0 + a_1x_2 + a_2y_2$$

$$q_5 = a_0 + a_1x_3 + a_2y_3$$

$$q_1 - q_3 = a_1(x_1 - x_2) + a_2(y_1 - y_2)$$

$$q_3 - q_5 = a_1(x_2 - x_3) + a_2(y_2 - y_3)$$

Difficulty to get a_1, a_2 & a_0

Shape function Method

No of nodes $= 3$ No of shape functions $= 3$ (N_1, N_2 & N_3)

$$N_1 + N_2 + N_3 = 1 \text{ ——(1)}$$

$$N_1x_1 + N_2x_2 + N_3x_3 = x \text{ ——(2)}$$

$$N_1y_1 + N_2y_2 + N_3y_3 = y \text{ ——(3)}$$

$$u = N_1q_1 + N_2q_3 + N_3q_5 \text{ ——(4)}$$

$$v = N_1q_2 + N_2q_4 + N_3q_6 \text{ ——(5)}$$

$$N_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}, \quad N_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}, \quad N_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}$$

From eq^{n}s (1), (2) & (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

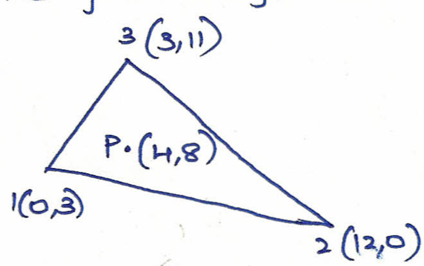
Area of Triangle $= A = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

$$A_1 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}$$

$$N_1 = \frac{A_1}{A}, \quad N_2 = \frac{A_2}{A}, \quad N_3 = \frac{A_3}{A}$$

$$N_1 + N_2 + N_3 = \frac{A_1 + A_2 + A_3}{A} = \frac{A}{A} = 1$$

Calculate the shape functions at (4,8) of the given triangle element. See the co-ordinates.



$$N_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 4 & 12 & 3 \\ 8 & 0 & 11 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 12 & 3 \\ 3 & 0 & 11 \end{vmatrix}}$$

$$N_1 = \frac{1(12\times 11 - 3\times 0) - 1(4\times 11 - 8\times 3) + 1(4\times 0 - 8\times 12)}{1(12\times 11 - 3\times 0) - 1(0\times 11 - 3\times 3) + 1(0\times 1 - 12\times 3)} = \frac{16}{105} = 0.152$$

$N_2 = 0.161$

$N_3 = 0.685$

Calculate the stiffness matrix for the triangular element in the previous problem . $t = 10mm$, $E = 2\times10^5 \, N/mm^2$, $\mu = 0.3$ (24)

$(x_1, y_1) = (0,3)$
$(x_2\, y_2) = (12,0)$
$(x_3\, y_3) = (3,11)$

$$K^{[e]} = \int_V B^T D B \, dv$$

$$[K^{[e]}]_{6\times6} = B^T_{6\times3} \, D_{3\times3} \, B_{3\times6} \, A^e t^e$$

$$\Rightarrow A^e = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{105}{2} = 52.5mm$$

$$[D]_{3\times3} \text{ consider in plain strain } = \frac{E}{(1-\mu)^2}\begin{bmatrix} \mu & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$[B]_{3\times6} = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix} = 105$$

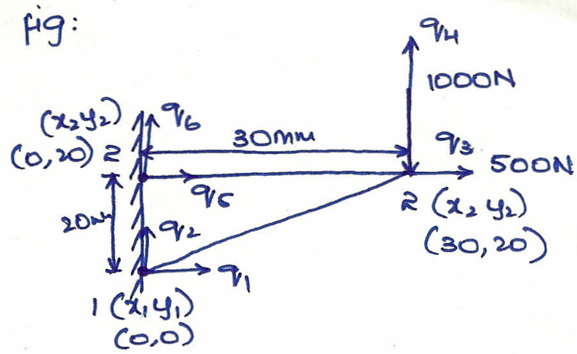
$$[K] = \frac{1}{105}\begin{bmatrix} -11 & 0 & -9 \\ 0 & -9 & -11 \\ 8 & 0 & -3 \\ 0 & -3 & 8 \\ 3 & 0 & 12 \\ 0 & 12 & 3 \end{bmatrix} \frac{2\times10^5}{0.49}\begin{bmatrix} 0.3 & 1 & 0 \\ 1 & 0.3 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \frac{1}{105}\begin{bmatrix} -11 & 0 & 8 & 0 & 3 & 0 \\ 0 & -9 & 0 & -3 & 0 & 12 \\ -9 & -11 & -3 & 8 & 12 & 3 \end{bmatrix}$$

$$[K] = \frac{1}{105}\,\frac{1}{105} \times \frac{2\times10^5}{0.49}\begin{bmatrix} -3.3 & -11 & -3.15 \\ -9 & -2.7 & -3.85 \\ 2.4 & 8 & -1.05 \\ -3 & -0.9 & 2.8 \\ 0.9 & 3 & 4.2 \\ 12 & 3.6 & 1.05 \end{bmatrix}\begin{bmatrix} -11 & 0 & 8 & 0 & 3 & 0 \\ 0 & -9 & 0 & -3 & 0 & 12 \\ -9 & -11 & -3 & 8 & 12 & 3 \end{bmatrix}$$

→ Determine the deflection at the point of load application for the two dimensional plane shown in fig:

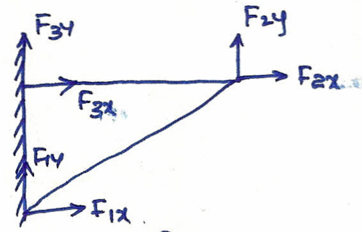
Load vector = [P]

$t = 10mm$
$E = 70 GPa$
$\mu = 0.3$



$$\{P\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ 500 \\ -1000 \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Nodal displacements $\{q\} = \begin{Bmatrix} q_1 = 0 \\ q_2 = 0 \\ q_3 \\ q_4 \\ q_5 = 0 \\ q_6 = 0 \end{Bmatrix}$



$$D = \frac{E}{(1-\mu)^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$K = \frac{10^6}{156} \begin{bmatrix} 31.5 & 0 & 0 & -21 & -31.5 & 21 \\ 0 & 90 & -18 & 0 & 18 & -90 \\ 0 & -18 & 40 & 0 & -40 & 18 \\ -21 & 0 & 0 & 14 & 21 & -14 \\ 31.5 & 18 & -40 & 21 & 71.5 & -39 \\ 21 & -90 & 18 & -14 & -39 & 104 \end{bmatrix}$$

$A = 300mm^2$ Take '10' common from the matrix

$$B = \frac{1}{60} \begin{bmatrix} 0 & 0 & 20 & 0 & -20 & 0 \\ 0 & -30 & 0 & 0 & 0 & 30 \\ -30 & 0 & 0 & 20 & 30 & -20 \end{bmatrix}$$

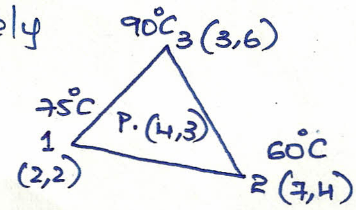
After elimination, Eliminate $R_1 C_1$ & $R_2 C_2$, $R_5 C_5$ & $R_6 C_6$ Because $q_1 = q_2 = q_5 = q_6 = 0$

$$\Rightarrow \frac{10^6}{156} \begin{bmatrix} 40 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{Bmatrix} 500 \\ -1000 \end{Bmatrix}$$

$$\Rightarrow \frac{10^6}{156}(40 q_3) = 500 \Rightarrow q_3 = 1.95 \times 10^{-3} mm$$

$$q_4 = -11.14 \times 10^{-3} mm$$

⇒ Derive the shape functions for a 3 noded triangular elements as shown in fig. And using the same determine the temperature as the point P(4,3) given that the temperature as the nodes 1,2,3 are 75°C, 90°C and 60°C respectively



N_1 N_2 N_3

Temp at point P (4,3)

$T(P) = N_1T_1 + N_2T_2 + N_3T_3 = 80^\circ C$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 7 & 3 \\ 2 & 4 & 6 \end{vmatrix}$$

$$A = \frac{1}{2}\left[1(42-12)-(12-6)+(8-14)\right]$$

$$= \frac{1}{2}(30-6-6) = \frac{18}{2} = 9$$

2. Calculate the value of Pr at the point A which is inside the 3 noded triangular element as shown in fig. The nodal values are $P_1 = 40\,MPa$, $P_2 = 34\,MPa$, $P_3 = 46\,MPa$. Point A is located at (2, 1.5). Assume Pr is linearly varying in the element

$P(A) = N_1P_1 + N_2P_2 + N_3P_3 = 39.4\,MPa$